18.726 Problem Set 10, due April 28

Do any *seven* problems, including all problems marked "Required". The reference for GAGA is J.-P. Serre, Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier (Grenoble)* **6** (1955-1956), 1–42.

This will be the last official problem set; note that you have two weeks for it. I may post one more for informational purposes, but you won't be required to turn it in.

1. Hartshorne III.4.5.
2. Hartshorne III.4.7.
3. Hartshorne III.5.1.
4. (Required) Hartshorne III.5.2.
5. Hartshorne III.5.7.
6. Hartshorne III.5.10.
7. Hartshorne V.2.15, parts (a) and (b) (yes, I really do mean the fifth chapter!).
8. Hartshorne III.5.3 (a) (for the definition of arithmetic genus) and V.3.1.
9. (Required) In this exercise, we classify vector bundles on $\mathbb{P}^1_k$, for k an algebraically closed field; this is due to (wait for it) Grothendieck, based on ideas of Serre. Throughout, let $\mathcal{E}$ denote a vector bundle (read: coherent locally free sheaf) on $\mathbb{P}^1_k$; let d denote the rank of $\mathcal{E}$, and define the *degree* of $\mathcal{E}$ as the unique integer n such that $\wedge^d \mathcal{E} \cong \mathcal{O}(n)$ (it exists and is unique by Corollary II.6.17). Aside: this actually is an exercise in Hartshorne, but in a strange place: it's V.2.6.

   (a) Suppose that
   $$0 \to \mathcal{O}(n_1) \to \mathcal{E} \to \mathcal{O}(n_2) \to 0$$
   is a short exact sequence of bundles and that $n_1 < n_2$. Prove that there exists a rank 1 subbundle of $\mathcal{E}$ of degree $> n_1$. (Hint: twist to reduce to the case $n_1 = -1$, then take cohomology.)

   (b) Suppose that
   $$0 \to \mathcal{O}(n_1) \to \mathcal{E} \to \mathcal{O}(n_2) \to 0$$
   is a short exact sequence of bundles and that $n_1 \geq n_2$. Prove that the exact sequence splits. (Hint: twist to reduce to the case $n_2 = 0$, then take cohomology again.)

(c) Prove that $\mathcal{E}$ contains a subbundle isomorphic to $\mathcal{O}(n)$ for some n; deduce that every vector bundle $\mathcal{E}$ admits a "composition series"

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_d = \mathcal{E}$$

by subbundles with $\mathcal{E}_i/\mathcal{E}_{i-1}$ locally free of rank 1 for $i = 1, \dots, d$. (Hint: every nontrivial vector bundle on $\mathbb{P}^1$ admits a rational section.)

(d) Prove that the set of degrees of rank 1 subbundles of $\mathcal{E}$ is bounded above. (Hint: compare to a filtration as in (a).)

(e) Prove that $\mathcal{E}$ admits a composition series with

$$\deg(\mathcal{E}_1/\mathcal{E}_0) \geq \deg(\mathcal{E}_2/\mathcal{E}_1) \geq \cdots \geq \deg(\mathcal{E}_d/\mathcal{E}_{d-1}).$$

(Hint: take $\mathcal{E}_1$ to have degree as large as possible.)

(f) Prove that $\mathcal{E} \cong \oplus_{i=1}^d \mathcal{O}(n_i)$ for some integers $n_1, \dots, n_d$.

10. (if you know some complex analysis) Let X be a scheme locally of finite type over $\mathbb{C}$. Prove that there exists a complex analytic space (locally ringed space looking locally like the zero locus of a set of holomorphic functions on an open subset of $\mathbb{C}^n$, equipped with the sheaf of holomorphic functions; you may take "holomorphic" to mean "locally defined by a convergent Taylor series") X^{an} and morphism $X^{\mathrm{an}} \to X$ which represents the functor taking each complex analytic space Y to $\mathrm{Hom}(Y, X)$ (that's homs of locally ringed spaces over $\mathrm{Spec}\,\mathbb{C}$). It's called the *analytification of* X. (Hint: since this is unique up to unique isomorphism, you can check for an affine variety, in which case it's what you think it is; the map to X comes from the adjointness property of Spec.)

11. (GAGA, first part) (Required, but see disclaimer in (a)) Put $X = \mathbb{P}^r_{\mathbb{C}}$; then in the previous problem, $X^{\mathrm{an}} = \mathbb{P}^{r,\mathrm{an}}_{\mathbb{C}}$ is the quotient of $\mathbb{C}^{r+1}$ by the action of $\mathbb{C}^*$; it can be made by glueing together $r+1$ copies of $\mathbb{C}^r$. Note that points of X^{an} correspond to closed points of X, so I will confuse these hereafter.

Prove that for any coherent sheaf $\mathcal{F}$ on X, the natural map

$$H^i(X, \mathcal{F}) \to H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

is an isomorphism, as follows. (Here $\mathcal{F}^{\mathrm{an}} = j^*(\mathcal{F})$ for $j : X^{\mathrm{an}} \to X$ the morphism coming from the previous problem.)

(a) First prove the claim for $\mathcal{F} = \mathcal{O}$. For H^0, you need the fact (at least for $n = 1$) that any bounded entire function on $\mathbb{C}^n$ is constant. For H^i with $i > 0$, you need a little analysis (the Dolbeault lemma); if you don't know what that is, skip this part and assume it for the later parts. (There might be a way to do this easily by finding a good Čech cover, in the sense of having all intersections being contractible, but I don't know one offhand. The standard cover of X has contractible opens but not contractible intersections.)

(b) Then prove the claim for $\mathcal{F} = \mathcal{O}(n)$ by induction on r. (Hint: use the setup from lecture, i.e., draw a hyperplane and form a short exact sequence of sheaves.)

(c) Prove the claim for any $\mathcal{F}$ using the fact that there is a surjection $\oplus_{i=1}^d \mathcal{O}(n_i) \to \mathcal{F}$ (from chapter II).

12. (GAGA, second part) Retain notation (and assume all results) from the previous problem.

(a) Prove that for any closed point $x \in X$, the stalk of the local ring of X^{an} at x is faithfully flat over the local ring of X at x. (Hint: the two local rings have the same completion.)

(b) Prove that forming the sheaf Hom between two coherent sheaves on X commutes with analytification. (Hint: use the flatness from (a).)

(c) Prove that the functor $\mathcal{F} \to \mathcal{F}^{\mathrm{an}}$ is fully faithful. (Hint: apply the previous problem and (b).)

13. (GAGA, third part) Retain notation (and assume all results) from the previous two problems. Prove that the functor $\mathcal{F} \to \mathcal{F}^{\mathrm{an}}$ is essentially surjective on coherent analytic sheaves on X^{an} (i.e., sheaves of $\mathcal{O}^{\mathrm{an}}$-modules which are locally cokernels of maps between finitely generated $\mathcal{O}^{\mathrm{an}}$-modules), as follows. Proceed by induction on r; at all times, assume the result for dimensions up to $r - 1$.

(a) Let $\mathcal{G}$ be a coherent analytic sheaf. Prove that for n sufficiently large, $\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{G}(n))$ is a nonincreasing function of n. Here $\mathcal{G}(n) = \mathcal{G} \otimes \mathcal{O}(n)^{\mathrm{an}}$. (Hint: draw a hyperplane H, set up the exact sequence

$$0 \to \mathcal{G}(-1) \to \mathcal{G} \to \mathcal{G}_H \to 0$$

as in lecture, take cohomology, and use the induction hypothesis.)

(b) Prove that for each closed point $x \in X$, there exists an integer n_0 such that for all $n \geq n_0$, $\mathcal{G}(n)$ is generated by global sections in a neighborhood of x (for the analytic topology). (Hint: it's enough to check on the stalk of x. Draw a hyperplane H through x, show that $H^0(X^{\mathrm{an}}, \mathcal{G}(n)) \to H^0(X^{\mathrm{an}}, \mathcal{G}_H(n))$ is surjective for n large, and note that $\mathcal{G}_H$ comes from a coherent algebraic sheaf on H by the induction hypothesis, so that we know by a Serre theorem that $\mathcal{G}_H(n)$ is generated by global sections for n large.)

(c) Construct a short exact sequence

$$\mathcal{F}_1^{\mathrm{an}} \to \mathcal{F}_2^{\mathrm{an}} \to \mathcal{G} \to 0,$$

where $\mathcal{F}_1, \mathcal{F}_2$ are coherent algebraic sheaves, and apply the previous problem to deduce that $\mathcal{G}$ is the analytification of a coherent algebraic sheaf. (Hint: from (b), some twist of $\mathcal{G}$ is generated by global sections, because X^{an} is compact. That gives a map $\mathcal{F}_2^{\mathrm{an}} \to \mathcal{G} \to 0$; then take the kernel of that map and do it again.)

(d) Deduce as corollaries that every projective complex analytic variety is algebraic, and any coherent analytic sheaf on same comes from a unique coherent algebraic sheaf. (Hint: push forward along a closed immersion into projective space to reduce to the corresponding statements about $\mathbb{P}^r$, which we already know.) Beware that there are compact complex analytic varieties which are not algebraic: for example, all complex tori (quotients of $\mathbb{C}^d$ by $\mathbb{Z}$-sublattices of rank $2d$) are compact complex analytic varieties, but for $d > 1$, many are not algebraic (the ones that are satisfy the so-called "Riemann conditions").