18.726 Problem Set 2, due Thursday, February 17

Please submit *eight* of the following problems, including all problems marked "Required", and at least one of the first two problems.

1. Let k be a field, and define the graded rings

$$S = k[x, y], \qquad S' = k[a, b, c, d]/(ac - b^2, ad - bc, bd - c^2),$$

in which each of x, y, a, b, c, d has degree 1.

   (a) Prove that $S \not\cong S'$.

   (b) In spite of (a), exhibit an isomorphism $\operatorname{Proj} S \cong \operatorname{Proj} S'$. (Hint: this is an example of a "rational normal curve".)

2. A *weighted projective space* over the field k is a scheme of the form $\operatorname{Proj} k[x_0, \ldots, x_n]$, where each x_i is homogeneous of some degree but *not necessarily 1*. Find an example of a weighted projective space which is not isomorphic to the ordinary projective space of the same dimension over the same field. (Hint: this is really a question about varieties, so you can use the fact that you know the difference between smooth and nonsmooth varieties. In particular, weighted projective spaces tend to be quite singular!)

3. Let R be a discrete valuation ring, and put $K = \operatorname{Frac}(R)$. Put $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \ldots, x_n]$, where each x_i has degree 1. Prove that the natural map $\mathbb{P}^n_R(R) \to \mathbb{P}^n_R(K)$ is a bijection.

4. Hartshorne II.2.15.

5. Hartshorne II.2.19.

6. (Required) Hartshorne II.3.6.

7. Hartshorne II.3.9.

8. (Required) Hartshorne II.3.10.

9. Hartshorne II.3.11.

10. Hartshorne II.3.12.

11. (Required) Hartshorne II.3.18.

12. Hartshorne II.4.12.