18.726 Problem Set 6, due Thursday, March 17

Please submit *six* of the following problems, including all problems marked "Required".

To give you some of the flavor of the theory of divisors on curves, I've included some problems from Chapter IV. For the purposes of this assignment, assume Theorem IV.1.3 (the Riemann-Roch theorem) in the following fashion: given a "curve" (here meaning a scheme correspondinng to a smooth, proper, irreducible variety of dimension 1) X over an algebraically closed field k, there exist a divisor K (the *canonical divisor*) and a nonnegative integer g (the *genus*) such that for all divisors D,

$$\dim_k(\Gamma(\mathcal{L}(D), X)) = \deg(D) + 1 - g + \dim_k(\Gamma(\mathcal{L}(K - D), X)).$$

(Note that you can deduce from this that X is projective; see problems.) Reminder: if you think of D as a Weil divisor, you can think of $\mathcal{L}(D)$ as the subsheaf of $\mathcal{K}_X$ whose sections over U are those rational functions f for which $(f) + D \geq 0$ on U (i.e., each point in U has a nonnegative coefficient in $(f) + D$).

1. Hartshorne II.6.2.
2. (Required) Hartshorne II.6.6.
3. Hartshorne II.7.5.
4. Hartshorne II.7.6.
5. (Required but easy) If a curve has genus g, what is the degree of its canonical divisor? (No, you don't need to know what a canonical divisor actually is to answer this question!)
6. Hartshorne IV.1.2.
7. Hartshorne IV.1.3.
8. Hartshorne IV.1.6.
9. Hartshorne IV.1.7.
10. Hartshorne IV.3.1.
11. Suppose X is a curve of genus 3. Prove that either X is hyperelliptic (see previous exercise) or X is isomorphic to a smooth plane curve of degree 4. (Hint: what does the complete linear system $|K|$ look like?) For the continuation of this discussion to genera 4 and 5, see the end of Chapter IV in Hartshorne, or better yet, see ACGH (the book by Arbarello, Cornalba, Griffiths and Harris; ignore the "Volume 1" label, there is no Volume 2).