18.726 Problem Set 7, due Thursday, March 31

Please submit *six* of the following problems, including all problems marked "Required". Conventions about "curves" are as on the previous problem set.

1. Hartshorne II.6.6.
2. Hartshorne II.7.4 (this is basically Genya's question from class on Tuesday).
3. Hartshorne II.7.11.
4. Hartshorne II.7.12.
5. Hartshorne II.8.5.
6. (Required) Put $A = k[x,y]/(y^2 - P(x))$, where k is a field and $P(x)$ is a polynomial of degree $2g+1$ with distinct roots. Verify that the cokernel of the map $d : A \to \Omega^1_{A/k}$ is $2g$-dimensional by showing that $\{x^i \, dx/y\}_{i=0}^{2g-1}$ constitutes a basis. (I.e., you are to compute the first "algebraic de Rham cohomology group" of a hyperelliptic curve.)
7. Prove Theorem III.7.14.1 (existence of residues) by any means you like, but make sure your proof works over an arbitrary field! (Hint: one approach is to adopt property (d) as the definition, then verify that its independence from the choice of uniformizing parameter amounts to a collection of polynomial identities with coefficients in $\mathbb{Z}$, so that they hold over any field iff they hold over $\mathbb{C}$.) The point is that later we'll give a statement of Riemann-Roch in terms of residues.
8. Hartshorne IV.4.2. ("Projectively normal" means that the homogeneous coordinate ring is an integrally closed domain.)
9. Hartshorne IV.5.6; if you need to, assume Hurwitz's theorem (Corollary IV.2.4).
10. Let X be the projective curve defined by an equation in $\mathbb{P}^2$ of the form $y^2 = P(x)$, where P is a cubic polynomial with distinct roots (i.e., the "elliptic curve" example from lecture). Verify explicitly that the Riemann-Roch theorem holds with $g = 1$ and $K = 0$. (Hint: use the structure of the class group that we worked out in lecture; that is, if you fix a closed point O, then every divisor of degree 0 is linearly equivalent to $(P) - (O)$ for some closed point P.)