# 18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
## Problem Set 1 (due Friday, February 13, in class)

Since I didn't do so earlier, let me take space here to enunciate the collaboration policy for this class. You may and *should* collaborate freely with others in the class in attempting to solve the problem sets. You may also consult outside references (textbooks, web sites, people not in the class, etc.). However, you must:

- write up your solutions independently;

- acknowledge all collaborators and outside references (a single list for the whole problem set will suffice). Exception: you need not acknowledge anything on the course web site (class notes, virtual office hours, etc.), Hartshorne's book, or me. (Also, you may not cite a statement in EGA which immediately implies the statement of a problem.)

Okay, back to work. Please submit exactly *eleven* of the following exercises, including all exercises marked "Required". (If you work in a group, you are encouraged not to submit the same subset of problems!)

1. (Required) Prove Yoneda's lemma as stated in lecture.

2. (Required) Describe a left adjoint of the forgetful functor on each of the following categories. (You don't need to justify your answers.)

   (a) Vector spaces over a field $K$.
   (b) Rings.
   (c) Topological spaces.

3. (Required) A *preadditive category* is a category in which each homset $\mathrm{Hom}(X, Y)$ is equipped with the structure of an abelian group in a functorial way, and composition is distributive over this group structure. For instance, there is an obvious way to view Ab as an additive category.

   (a) Reformulate this definition rigorously using a natural transformation.
   (b) Define a *kernel* and *cokernel* of a morphism $X \to Y$ in a preadditive category. (Hint: each of these consists of an *object* plus a *morphism* satisfying a certain universal property.)
   (c) Show that for any field $K$, if we take the category of finitely generated modules over $K[x]$, and the full subcategory of finite *free* modules over $K[x]$, then both categories admit cokernels, but the image of a cokernel under the inclusion functor may not be a cokernel.
   (d) You might think that if a category admits a faithful functor to Ab, then this gives it the structure of a preadditive category. Give a familiar example where this fails.

We will use this exercise later to define the concept of an *abelian category*.

4. (Required) Here is a variant on the product which we will be using extensively later. Given two morphisms $Y \to X$ and $Z \to X$ in a category, a *fibre product* of these two morphisms is a limit of the diagram
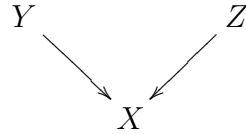


We usually denote a fibre product by $Y \times_X Z$.

   (a) Show that fibre products exist in the category of sets, and describe them explicitly.

   (b) Prove that $X \to Y$ is a monomorphism if and only if the induced map $X \to X \times_Y X$ is an isomorphism.

   (c) Suppose $X \to Y$ is a monomorphism, and let $W \to X$ and $Z \to X$ be two other morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic.

5. (a) Prove that inverse limits exist in the category of topological spaces. (Hint: the underlying set of the inverse limit is the same as the inverse limit of the underlying sets. Figure out what the right topology should be.)

   (b) Prove that an inverse limit of compact topological spaces is compact. You may use without proof *Tikhonov's theorem*: any product of compact topological spaces is again compact.

6. (Required) Hartshorne II.1.2.

7. Hartshorne II.1.4.

8. (Required) Hartshorne II.1.8.

9. Hartshorne II.1.10 and II.1.12 (they count together as one problem).

10. Hartshorne II.1.11.

11. Hartshorne II.1.13.

12. Hartshorne II.1.14.

13. Hartshorne II.1.16.

14. Hartshorne II.1.17.

15. (Required) Hartshorne II.1.18.

16. Hartshorne II.1.19.