18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 10 (due Friday, April 24, in class)

Please submit *nine* of the following exercises (counting with multiplicities as indicated), including all items marked "Required". More exercises than usual this week are from Eisenbud-Harris; let me know if you need access to a copy.

1. Recall that a *paracompact* topological space is a Hausdorff space on which every open covering admits a locally finite refinement. Prove that if X is paracompact, then for every locally finite open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, there exists another open covering $\mathfrak{V} = \{V_i\}_{i \in I}$ of X with the same index set, such that for each $i \in I$, the closure of V_i in X is contained in U_i. (I include this mostly so that you may assume it for the next exercise. See Bourbaki's *Topologie generale*.)

2. Let X be a paracompact topological space.

   (a) Let $\mathfrak{U}$, $\mathfrak{V}$ be as in the previous exercise. Let $\mathcal{F} \to \mathcal{G}$ be a surjective morphism of sheaves of abelian groups on X. Prove that for any open subset T of X, any element $s \in \Gamma(\check{C}^i(\mathfrak{U}, \mathcal{G}), T)$ lifts to $\Gamma(\check{C}^i(\mathfrak{W}, \mathcal{F}), T)$ for some refinement $\mathfrak{W}$ of $\mathfrak{U}$. (Hint: for each $x \in X$, one can find an open neighborhood W_x of x meeting only finitely many of the U_i. Show that you can choose W_x so that $x \in U_i$ implies $W_x \subseteq U_i$, $x \in V_i$ implies $W_x \subseteq V_i$, $W_x \cap V_i \neq \emptyset$ implies $x \in U_i$, and $x \in T$ implies s lifts to W_x.)

   (b) Use this to show that the functors

   $$\mathcal{F} \mapsto \varinjlim_{\mathfrak{U}} \check{C}^{\cdot}(\mathfrak{U}, \mathcal{F})$$

   are exact, and so conclude that Čech cohomology and sheaf cohomology coincide on a paracompact space.

3. Hartshorne III.4.10.

4. Hartshorne III.5.1.

5. (Counts as two) Hartshorne III.5.8.

6. (Required) Hartshorne III.5.10.

7. (Counts as two) In this exercise, we classify vector bundles on $\mathbb{P}^1_k$, for k an algebraically closed field; this is due to Grothendieck, based on ideas of Serre. Throughout, let $\mathcal{F}$ denote a finitely generated locally free quasicoherent sheaf on $\mathbb{P}^1_k$; let d denote the rank of $\mathcal{F}$, and define the *degree* of $\mathcal{F}$ as the unique integer n such that $\wedge^d \mathcal{F} \cong \mathcal{O}(n)$; this exists and is unique by Corollary II.6.17. (Compare Hartshorne exercise V.2.6 (sic).)

(a) Suppose that

$$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

is a short exact sequence of quasicoherent sheaves and that $n_1 < n_2$. Prove that there exists a rank 1 subbundle of $\mathcal{F}$ of degree $> n_1$. (Hint: twist to reduce to the case $n_1 = -1$, then take cohomology.)

(b) Suppose that

$$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

is a short exact sequence of quasicoherent sheaves and that $n_1 \geq n_2$. Prove that the exact sequence splits. (Hint: twist to reduce to the case $n_2 = 0$, then take cohomology again.)

(c) Prove that $\mathcal{F}$ contains a subsheaf isomorphic to $\mathcal{O}(n)$ for some n; deduce that $\mathcal{F}$ admits a "composition series"

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d = \mathcal{F}$$

by subsheaves with $\mathcal{F}_i/\mathcal{F}_{i-1}$ locally free of rank 1 for $i = 1, \dots, d$. (Hint: if $\mathcal{F}$ is nonzero, then it admits a rational section.)

(d) Prove that the set of degrees of rank 1 subsheaves of $\mathcal{F}$ is bounded above. (Hint: compare to a filtration as in (c).)

(e) Prove that $\mathcal{F}$ admits a composition series with

$$\deg(\mathcal{F}_1/\mathcal{F}_0) \geq \deg(\mathcal{F}_2/\mathcal{F}_1) \geq \cdots \geq \deg(\mathcal{F}_d/\mathcal{F}_{d-1}).$$

(Hint: take $\mathcal{F}_1$ to be a subsheaf of $\mathcal{F}$ with degree as large as possible.)

(f) Prove that $\mathcal{F} \cong \oplus_{i=1}^d \mathcal{O}(n_i)$ for some integers $n_1, \dots, n_d$.

8. (Required) Let X be a nonempty closed subscheme of $\mathbb{P}^r_k$, for k an algebraically closed field. Prove that for a generic hyperplane H, we have $\dim(X \cap H) < \dim(X)$; that is, the hyperplanes H for which this fails correspond to the points of a closed subscheme of the Grassmannian. (This Grassmannian is itself a projective space, in the coefficients describing H in terms of $x_0, \dots, x_n$.)

9. (Eisenbud-Harris III-58) Let A be a noetherian ring. Let X be a closed subscheme of $\mathbb{P}^r_A$ for some $r \geq 1$. Prove that for any nonnegative integer n, the function

$$t \mapsto \dim_{\kappa(t)} \Gamma(X_t, \mathcal{O}(n))$$

is upper semicontinuous; that is, for each $m \in \mathbb{Z}$, the set of points where the function has value at least m is closed in $\operatorname{Spec} A$.

10. Eisenbud-Harris III-60.

11. (Required) The *Hilbert function* of a closed subscheme X of a projective space over a field k is the function on nonnegative integers defined by $n \mapsto \dim_k H^0(X, \mathcal{O}(n))$.

(a) Find the Hilbert polynomial and the Hilbert function of all subschemes of the plane of length 3 over an algebraically closed field.

(b) Give an example of two schemes with the same Hilbert polynomial but not the same Hilbert function.

12. Eisenbud-Harris III-66.

13. Check the numerical criterion for flatness explicitly for Hartshorne Example III.9.8.4.