18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 11 (due Friday, May 1, in class)

Please submit *eight* of the following exercises (counting with multiplicities as indicated), including all items marked "Required".

1. Hartshorne III.2.4. (This depends on Hartshorne III.2.3, which was a nonrequired problem previously.)

2. Hartshorne III.4.6.

3. Hartshorne III.9.8.

4. Hartshorne III.9.9 (you may assume the previous exercise for this even if you don't turn it in).

5. (Required)

   (a) I gave the wrong definition of coherent sheaves in lecture. The correct one is the following. Let $\mathcal{F}$ be a sheaf on a ringed space $(X, \mathcal{O}_X)$. Then $\mathcal{F}$ is coherent if for any open subset U of X and any morphism $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}$, *not necessarily surjective*, the kernel of ϕ is finitely generated.

   (b) Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be an exact sequence of quasicoherent sheaves on X. Prove that if any two of $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ are coherent, then so is the third.

6. Compute the Čech cohomology of the twisting sheaves $\mathcal{O}(n)$ on the analytic projective space $\tilde{\mathbb{P}}^r_{\mathbb{C}}$ using the usual cover by $r+1$ affine spaces.

7. (Required) In Serre's finiteness theorem, suppose that we don't assume that the ring A is noetherian, but we do assume that $A[x_1, \dots, x_n]$ is coherent as a module over itself for each nonnegative integer n, and the sheaf $\mathcal{F}$ is coherent. Prove that the conclusion of Serre's theorem still holds. (Hint: prove that the $H^i(X, \mathcal{F})$ are coherent A-modules.)

8. (Counts as two) Let X be a proper scheme over $\mathbb{C}$, with analytification $h : \tilde{X} \to X$. Using Chow's lemma to reduce to the projective case, show that GAGA still applies to X in the following senses.

   (a) Any coherent sheaf on $\tilde{X}$ is the pullback of a unique coherent sheaf on X.

   (b) If $\mathcal{F}, \mathcal{G}$ are coherent sheaves on X, then any morphism $h^*\mathcal{F} \to h^*\mathcal{G}$ of coherent sheaves on $\tilde{X}$ is the pullback of a unique morphism $\mathcal{F} \to \mathcal{G}$ of coherent sheaves on X.

   (c) For any coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$ for all $i \geq 0$.

9. (a) Formulate an analogue of Grothendieck's analytification theorem, giving a "varietification" functor from reduced separated schemes of finite type over an algebraically closed field k to abstract algebraic varieties over k.

   (b) Check the assertion of (a) explicitly for $\mathbb{A}^1_k$. (You might want to imagine the rest of the proof of the theorem you stated in (a), but you don't have to turn it in.)

10. Let $j : X \to \mathbb{P}^r_{\mathbb{C}}$ be an immersion. Prove that the following are equivalent.

    (a) X^{an} is compact.

    (b) j is a closed immersion.

    (c) $X \to \operatorname{Spec} \mathbb{C}$ is proper.

11. Read the handout on spectral sequences, then verify that the construction of the derived exact couple does indeed give an exact couple.

12. Read the handout on spectral sequences, then write out explicitly what the effect of the differential d_r is on $E_r^{p,q}$. You don't have to check that your recipe is well-defined, as long as you indicate how your answer agrees with the general construction (which is already well-defined given the previous exercise). You might want to check Bott and Tu, but I'd beware of the signs if I were you.