18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 12 (due Friday, May 8, in class)

Please submit *eight* of the following exercises, including all items marked "Required".

1. Do PS 8, problem 9 if you didn't submit it then. (A related problem is Hartshorne III.6.1.)

2. Hartshorne III.6.2.

3. Hartshorne III.6.3.

4. Hartshorne III.6.6.

5. Hartshorne III.7.2(a).

6. Hartshorne III.8.1 and III.8.2. (You may use spectral sequences on III.8.1 if you wish.)

7. (Required) Let A be a ring, and put $X = \mathbb{P}^n_A$ for some $n \geq 1$. For each integer $p \in \{0, \dots, n\}$ and each $q \in \mathbb{Z}$, prove that $H^q(X, \Omega^p_{X/A})$ is a finite free A-module, and compute its rank. (Hint: see Hartshorne III.7.3 for part of the answer. Bigger hint: remember that $\Omega_{X/A} \cong \mathcal{O}_X(-n-1)$.)

8. Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories such that $\mathcal{C}_1$ has enough injectives. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact additive functor. For $C^\cdot$ a cohomologically graded complex in nonnegative degrees, we define the right derived functors $R^iF(C^\cdot)$ by constructing a quasi-isomorphism $C^\cdot \to I^\cdot$ to a complex of injectives and putting $R^iF(C^\cdot) = h^i(F(I^\cdot))$. Prove that this is well-defined and functorial. (Hint: the existence of $C^\cdot \to I^\cdot$ was a previous exercise. Using a pushout construction, you can reduce well-definedness to comparing the results of using two injective complexes using a map $I^\cdot \to J^\cdot$. Similarly for functoriality.)

9. For $F = \Gamma : \underline{\mathrm{Sh}_{\mathrm{Ab}}}(X) \to \underline{\mathrm{Ab}}$ the global sections functor, the derived functors in the previous exercise are called the *hypercohomology* of a complex of sheaves $\mathcal{F}^\cdot$, denoted $\mathbb{H}^i(X, \mathcal{F}^\cdot)$. Prove that, for any cover $\mathfrak{U}$ which is good for each of the $\mathcal{F}^\cdot$, $\mathbb{H}^i(X, \mathcal{F}^\cdot)$ is isomorphic to the i-th cohomology of the total complex associated to the double complex $\check{C}^\cdot(\mathfrak{U}, \mathcal{F}^\cdot)$. For instance, if $\mathcal{F}^\cdot$ consists of $0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$ and $\mathfrak{U} = \{U_1, U_2\}$, then $\mathbb{H}^i(X, \mathcal{F}^\cdot)$ is the cohomology of the complex

$$0 \to \mathcal{F}^0(U_1) \oplus \mathcal{F}^0(U_2) \to \mathcal{F}^0(U_1 \cap U_2) \oplus \mathcal{F}^1(U_1) \oplus \mathcal{F}^1(U_2) \to \mathcal{F}^1(U_1 \cap U_2) \to 0$$

in which the first arrow carries (f_1, f_2) to $(f_1 - f_2, d(f_1), d(f_2))$ and the second arrow carries (f, ω_1, ω_2) to $d(f) - \omega_2 + \omega_1$. (See the spectral sequence handout for the general definition of the total complex associated to a double complex.)

10. (Required) Let k be a field of characteristic zero and put $X = \mathbb{P}^n_k$ for some $n \geq 1$. Use the previous exercises to show that

$$\mathbb{H}^i(X, \Omega^{\cdot}_{X/k}) = \begin{cases} k & i = 0, 2, \dots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this computes the right Betti numbers; this is a special case of a theorem of Grothendieck (whose proof uses GAGA). Also in particular, even though the scheme is only n-dimensional, and the complex only goes up to degree n, you get a nonzero contribution in hypercohomology in degree $2n$. Optional: replace k by an arbitrary ring and derive a similar result.

11. Let k be a field of characteristic zero, and let $P(x) \in k[x]$ be a polynomial of degree $2g+1$ with no repeated roots. Let X be the (smooth projective) hyperelliptic curve defined by the affine equation $y^2 = P(x)$. Using results from previous exercises, prove that

$$\dim_k \mathbb{H}^i(X, \Omega^{\cdot}_{X/k}) = \begin{cases} 1 & i = 0 \\ 2g & i = 1 \\ 1 & i = 2 \\ 0 & i > 2. \end{cases}$$

Again, this computes the expected Betti numbers.

12. Prove Hartshorne Theorem III.7.14.1 as follows. For k a field, define the function $\mathrm{Res} : k((t))\,dt \to k$ sending $\sum_{i \in \mathbb{Z}} a_i t^i \, dt$ to a_{-1}. For $b \in tk[\![t]\!] - t^2k[\![t]\!]$, define $s_b : k((t)) \to k((t))$ to be the substitution map

$$\sum_{i \in \mathbb{Z}} a_i t^i \mapsto \sum_{i \in \mathbb{Z}} a_i b^i,$$

and let $ds_b : k((t))\,dt \to k((t))\,dt$ be the map

$$f\,dt \mapsto s_b(f) \frac{db}{dt}\,dt.$$

   (a) Prove that for each positive integer m, the composition $\mathrm{Res} \circ ds_b : t^{-m}k[\![t]\!]\,dt \to k$ can be written as a polynomial $P_m(a_{-1}, \dots, a_{-m}, b_1, \dots, b_m)$ with coefficients in $\mathbb{Z}$, not depending on k.

   (b) Use (a) to prove that for $k = \mathbb{C}$, $\mathrm{Res} \circ ds_b = \mathrm{Res}$ for all b. (Hint: one method uses the Cauchy integral formula. Another method involves computing the cokernel of the map $\frac{d}{dt}$.)

   (c) Use (a) and (b) to prove that for any field k, $\mathrm{Res} \circ ds_b = \mathrm{Res}$.

   (c) Use (c) to prove Theorem III.7.14.1.

13. (Required) Let $I^\cdot$ be a cohomologically graded complex in nonnegative degrees consisting of injective objects (in some abelian category) which has no cohomology in degrees $0, \dots, r-1$. Prove that we can split $I^\cdot$ as a direct sum of two complexes of injective objects $I_1^\cdot \oplus I_2^\cdot$, such that $I_1^\cdot$ is exact, $I_1^i = 0$ for $i > r$, and $I_2^i = 0$ for $i < r$. (Hint: first check the case $r = 1$, then induct on r.)