18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 2 (due Friday, February 20, in class)

Please submit exactly *twelve* of the following exercises, including all exercises marked “Required”.

1. (Required) Prove that <u>Set</u> and its opposite category are not equivalent, by finding an arrow-theoretic property satisfied by <u>Set</u> but not by its opposite, or vice versa. (Many solutions are possible.)

2. Prove that the sheafification functor from presheaves on a topological space X to sheaves on X, and the forgetful functor from sheaves to presheaves, form an adjoint pair.

3. (Required)

   (a) Complete the proof of the basis lemma (“sheaves” handout) in the case of a nice basis.

   (b) Complete the proof of the basis lemma in general.

4. Let $i : Z \hookrightarrow X$ be an inclusion of topological spaces in which Z carries the subspace topology. Let $\mathcal{F}$ be a sheaf on X.

   (a) If Z is open, prove that $i^{-1}\mathcal{F}$ may be canonically identified with the restriction $\mathcal{F}|_Z$.

   (b) If $Z = \{x\}$, prove that $i^{-1}\mathcal{F}$ may be canonically identified with the stalk $\mathcal{F}_x$.

5. (Required if you’ve never done it before) Prove the five lemma (“abelian sheaves” handout).

6. (Required if you’ve never done it before)

   (a) Complete the proof of the snake lemma (“abelian sheaves” handout).

   (b) Let

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r] \arrow[d, "f_0"] & B_0 \arrow[r] \arrow[d, "g_0"] & C_0 \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f_1"] & B_1 \arrow[r] \arrow[d, "g_1"] & C_1 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d, "f_2"] & B_2 \arrow[r] \arrow[d, "g_2"] & C_2 \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & A_3 \arrow[r] & B_3 \arrow[r] & C_3 \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in which the rows are exact and the columns are complexes. Use the snake lemma to show that

$$\frac{\ker(f_1)}{\mathrm{im}(f_0)} \to \frac{\ker(g_1)}{\mathrm{im}(g_0)} \to \frac{\ker(h_1)}{\mathrm{im}(h_0)} \xrightarrow{\delta} \frac{\ker(f_2)}{\mathrm{im}(f_1)} \to \frac{\ker(g_2)}{\mathrm{im}(g_1)} \to \frac{\ker(h_2)}{\mathrm{im}(h_1)}$$

is exact, where δ is defined as in the snake lemma, and the other maps are induced naturally by the horizontal arrows. (This will later give us the *long exact sequence in homology*.)

7. (a) Let $f^* : \mathcal{C}_1 \to \mathcal{C}_2$ and $f_* : \mathcal{C}_2 \to \mathcal{C}_2$ be an adjoint pair of functors, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are abelian categories. Prove that f^* is right exact and f_* is left exact.

   (b) Use this to prove that as functors from $\underline{\mathrm{Mod}}_R$ to itself (for any given ring R), $\mathrm{Hom}(X, \cdot)$ is left exact and $X \otimes_R \cdot$ is right exact. (I.e., prove that these two form an adjoint pair.)

8. The Taylor series map gives a ring homomorphism from the germs at 0 of the sheaf of holomorphic functions on $\mathbb{C}^n$ to $\mathbb{C}[\![x_1, \ldots, x_n]\!]$. Prove that this map is injective, and that its image consists of the power series which converge on some open polydisc around 0.

9. Do Hartshorne II.1.3, but for (b), instead of the example given in class, make an example on the space $\{1, 2, 3\}$ with topology generated by $\{1, 2\}, \{2, 3\}$.

10. (Required)

    (a) Prove Hartshorne II.1.7 by checking that formation of kernel, image, and cokernel of a morphism of abelian sheaves commute with passage to stalks.

    (b) Prove that abelian sheaves on a fixed topological space form an abelian category in which the categorical kernel, image, and cokernel coincide with the notions we defined.

11. Prove that a preadditive category with finite products is additive, that is, the product can be naturally equipped with a biproduct structure.

12. A topological space is *quasicompact* if every open cover admits a finite subcover. (The term *compact* is usually reserved for a space which is not just quasicompact but also Hausdorff.)

    (a) Recall that for any ring R, $\mathrm{Spec}(R)$ is quasicompact. (We did this in class, so you don't have to write anything here.)

    (b) Prove that if R is a noetherian ring, then *any* subset of $\mathrm{Spec}(R)$ is quasicompact.

    (c) Prove that (b) can fail if R is not noetherian. (Hint: construct a ring in which $\mathrm{Spec}(R)$ contains an infinite subset carrying the discrete topology.)

13. (Required) Describe $\operatorname{Spec}\mathbb{Z}$ by listing:
    (a) the points;
    (b) the open sets;
    (c) the sections of the structure sheaf over each open set.

14. (a) What is the maximum number of elements of $\mathbb{Z}$ which do not have the same value at any point of $\operatorname{Spec}(\mathbb{Z})$?
    (b) Same question with $\mathbb{Z}$ replaced by the ring $\mathbb{Z}[i]$ of Gaussian integers?

15. (Required) Hartshorne II.2.1 and II.2.2 (these count as one exercise).

16. This exercise is an arithmetic analogue of the fact that the complement of the origin in $\mathbb{A}^2$ is not an affine algebraic variety. Let X be the locally ringed space obtained from $\operatorname{Spec}\mathbb{Z}[x]$ by removing the point $(2, x)$.
    (a) Prove that $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}[x]$. (Hint: cover X with the distinguished opens $D(2)$ and $D(x)$ of $\operatorname{Spec}\mathbb{Z}[x]$.)
    (b) Use (a) to show that X is not affine.

17. Hartshorne II.2.10 *or* Hartshorne II.2.11 (but not both).