18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 3 (due Friday, February 27, in class)

Please submit exactly *thirteen* of the following exercises, including all exercises marked “Required”.

1. (Required) Use the fact that the structure sheaf on $\mathrm{Spec}(R)$ is a sheaf to recover the Chinese remainder theorem in R.

2. Let S be any set. Prove that $X = \mathrm{Spec}\,\mathbb{F}_2^S$ (the product of copies of $\mathbb{F}_2$ indexed by S) can be equipped with a map $f : S \to X$ such that for any function $g : S \to Y$ with Y a compact topological space, there exists a unique continuous map $h : X \to Y$ with $g = h \circ f$. In other words, X is the *Stone-Čech compactification* of the discrete topological space on S.

3. (for those who know about elliptic curves) Let k be an algebraically closed field. Let E be an elliptic curve over k. Let $P, Q \in E(k)$ be two points such that the difference $P - Q$ is not a torsion point under the group law on $E(k)$. Prove that $E \setminus \{P, Q\}$ is an open subscheme of E which is affine but not distinguished.

4. Hartshorne II.2.3.

5. (Required) Read the discussion of graded rings and Proj, then do the following.
   (a) Show that for a ring A, my definition of $\mathbb{P}^n_A$ is canonically isomorphic to Hartshorne's definition of it as $\mathrm{Proj}\,A[x_0, \dots, x_n]$.
   (b) Do parts (a)-(c) of Hartshorne II.2.14.

6. Let k be a field, and define the graded rings

$$S = k[x, y], \qquad S' = k[a, b, c, d]/(ac - b^2, ad - bc, bd - c^2)$$

   in which each of x, y, a, b, c, d is homogeneous of degree 1. Prove that $S \not\cong S'$, but $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$. (Hint: look up the notion of a *rational normal curve*.)

7. (a) Construct a morphism $\mathrm{Spec}\,R \to \mathbb{P}^n_{\mathbb{Z}}$ for some ring R, whose image does not lie in a distinguished open subset.
   (b) Let R be a discrete valuation ring with fraction field K. Prove that the natural map $\mathbb{P}^n_{\mathbb{Z}}(R) \to \mathbb{P}^n_{\mathbb{Z}}(K)$ is a bijection.

   (You may use either my or Hartshorne's definition of $\mathbb{P}^n$ for these.)

8. Hartshorne II.2.15.

9. (Required) Hartshorne II.2.16 and II.2.17 (these count as one exercise).

10. Hartshorne II.2.18.

11. Hartshorne II.2.19.

12. Verify that for $Y \to X$ and $Z \to X$ morphisms of schemes, the fibre product $Y \times_X Z$ in the category of schemes is also a fibre product in the category of locally ringed spaces. (Hint: imitate the construction of the fibre product, starting with the case where X, Y, Z are all affine.)

13. Which of these properties of a morphism of schemes is stable under arbitrary base change?

    (a) injectivity (on points)

    (b) surjectivity (on points)

    (c) bijectivity (on points)

14. (Required) Hartshorne II.3.9.

15. Hartshorne II.3.10.

16. (Required) Here is a device we will use over and over again to construct properties of morphisms of schemes. (I'll need a name for this; call it the *weak collater*.) Let P be a property of morphisms of schemes $f : Y \to X$ which is only defined when X is affine. Suppose that the following condition holds.

    (i) Let $f : Y \to X$ be a morphism with X affine. Let $D(g_1), \dots, D(g_n)$ be a finite covering of X by distinguished open subsets. Then f has property P if and only if the induced morphisms $Y \times_X D(g_i) \to D(g_i)$ all have property P.

    Prove that for $f : Y \to X$ an arbitrary morphism, the following are equivalent.

    (a) For a *single* open affine cover $\{U_i\}_{i \in I}$ of X, each induced morphism $Y \times_X U_i \to U_i$ has property P.

    (b) For *each* open affine cover $\{U_i\}_{i \in I}$ of X, each induced morphism $Y \times_X U_i \to U_i$ has property P. (In other words, for every open affine $U \subseteq X$, the induced morphism $Y \times_X U \to U$ has property P.)

    (If these hold, we say that f also has property P. A property constructed this way is automatically *local on the target*.)

17. (Required) Add the following hypothesis to the previous exercise. (I'll call this the *strong collater*.)

    (ii) Let $f : Y \to X$ be a morphism with X affine, having property P. Then for any morphism $g : Z \to X$ with Z also affine, $f \times g : Y \times_X Z \to Z$ has property P.

    Then deduce that (a) and (b) in the previous exercise are equivalent to this condition.

(c) For every morphism $g : Z \to X$ with Z affine, $f \times g : Y \times_X Z \to Z$ has property P.

Then prove also that property P (for an arbitrary morphism) is stable under *arbitrary* base change. Almost all properties of morphisms that we will consider are both local on the target and stable under base change.

18. (Suggested by Kaloyan) Give an alternate proof of the (first) fundamental theorem of affine schemes, as follows. Let R be a ring and let M be an R-module.

    (a) Put $M' = \Gamma(\tilde{M}^+, \operatorname{Spec}(R))$. (I leave it to you to define the sheafification of a presheaf specified on a basis.) Describe the natural R-module structure on M' and the natural R-module homomorphism $M \to M'$.

    (b) Show that for each prime ideal $\mathfrak{p}$, the induced map $M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is an isomorphism.

    (c) Deduce that $M \to M'$ is an isomorphism. (Hint: consider the annihilators of the kernel and cokernel of the map.)

19. Let L/K be a field extension of degree d. Prove that $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L$ has at most d points, with equality if and only if L is Galois over K.

20. (Required) Let $X \to S$ be a morphism of schemes. We say X is a *group scheme over* S if for each S-scheme $Y \to S$, the set $X_S(Y)$ of Y-valued points of $X \to S$ (i.e., maps $Y \to X$ commuting with the other two) comes equipped with a group structure which is functorial in Y.

    (a) Write this definition in terms of natural transformations.

    (b) Write this definition directly in terms of $X \to S$.

    (c) Describe explicitly a group scheme $\mathbb{G}_a$ over $\operatorname{Spec} \mathbb{Z}$ such that for any ring R, $\mathbb{G}_a(R)$ is the additive group of R.

    (d) Describe explicitly a group scheme $\mathbb{G}_m$ over $\operatorname{Spec} \mathbb{Z}$ such that for any ring R, $\mathbb{G}_m(R)$ is the multiplicative group of units of R.