18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 4 (due Friday, March 6, in class)

Please submit exactly *twelve* of the following exercises, including all exercises marked "Required".

1. Eisenbud-Harris II-11 and II-20. (If you need more examples, I recommend some of the other exercises in Eisenbud-Harris, chapter II.)

2. Here is a slightly stronger version of the third fundamental theorem of affine schemes. Let A be a ring and let V be a quasicompact open subset of Spec(A). Prove that every quasicoherent sheaf on $\mathcal{O}_{\mathrm{Spec}(A)}|_V$ has the form $\tilde{M}|_V$ for some A-module M. (If you get stuck, see EGA 1, Théorème 1.4.1.)

3. (a) Prove that a morphism $f : Y \to X$ is affine if and only if it occurs as the relative Spec of a quasicoherent sheaf of $\mathcal{O}_X$-modules.

   (b) Prove that a morphism $f : Y \to X$ is finite if and only if it occurs as the relative Spec of a quasicoherent sheaf of $\mathcal{O}_X$-modules which is locally finite over $\mathcal{O}_X$.

   (Hint for both: view $f_*\mathcal{O}_Y$ as an $\mathcal{O}_X$-module via $f^\sharp$.)

4. (Required) Hartshorne II.4.1.

5. Hartshorne II.4.2.

6. (Required) Hartshorne II.4.3.

7. Hartshorne II.4.6.

8. (Required) Hartshorne II.4.8. (For the record, this is EGA Remarque 1.5.5.12.)

9. (Not required, but strongly recommended) Let $f : X \to Y$ be a continuous map of topological spaces. We say f is *closed* if the image of every closed subset of X is a closed subset of Y. We say that f is *proper* if for any topological space Z, the induced map $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed.

   (a) Show that if f is injective, closed implies proper. Then exhibit an example to illustrate that closed may not imply proper in general.

   (b) Show that properness is local on the target, i.e., if $\{U_i\}$ is a cover of Y by open subsets, and $f^{-1}(U_i) \to U_i$ is proper for each i, then f is proper.

   (c) Show that f is proper if and only if f is closed and for each $y \in Y$, $f^{-1}(y)$ is quasicompact.

   (d) Suppose X is Hausdorff and Y is locally compact. Show that f is proper if and only if for every compact subset K of Y, $f^{-1}(K)$ is compact.

   (For more along these lines, see Bourbaki, *Topologie Generale*, §1.6.)

10. Exhibit an example of a scheme X over $\operatorname{Spec} \mathbb{F}_p(x)$ which is reduced, but for which $X \times_{\operatorname{Spec} \mathbb{F}_p(x)} \operatorname{Spec} \mathbb{F}_p(x)^{\mathrm{alg}}$ is not reduced (i.e., X is not *geometrically reduced*).

11. (Required) Show that each of the following properties of morphisms satisfies the axioms of the strong collater, when defined in the naïve way over an affine base.

    (a) Locally of finite type.

    (b) Locally of finite presentation.

12. (Required)

    (a) A morphism $f : Y \to X$ is *quasicompact* if the underlying topological space of Y is quasicompact (every open cover has a finite subcover). Show that this notion satisies the axioms of the strong collater, so that we obtain a notion of quasicompactness of morphisms which is local on the base and stable under base change. (This is essentially Hartshorne II.3.2.)

    (b) Prove that any morphism which is affine is quasicompact.

    (c) Prove that a morphism is of finite type if and only if it is locally of finite type and quasicompact. (This is Hartshorne II.3.3(a). The other parts of that exercise should be obvious by now.)

13. Prove that a composition of morphisms of finite presentation is again of finite presentation. This is tricky because a closed immersion need not be of finite presentation (unless the target is locally noetherian).

14. A morphism $f : Y \to X$ is *quasiseparated* if the diagonal $\Delta : Y \to Y \times_X Y$ is finite (not necessarily a closed immersion). Prove that this property is local on the base and stable under base change.

15. Prove that the category of abstract algebraic varieties over the algebraically closed field k is equivalent to the category of schemes which are reduced and locally of finite type over $\operatorname{Spec}(k)$. (Hint: most of this proof is written out in Hartshorne.)

16. Hartshorne II.5.1.

17. (Required) Describe the diagonal $\Delta : \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ explicitly, and deduce that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is separated.