18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 5 (due Friday, March 13, in class)

Please submit exactly *eleven* of the following exercises, including all exercises marked "Required". In case it comes up, you may use Hartshorne's more restrictive definition of projectivity in place of mine.

1. This is to make up for the incorrect Eisenbud-Harris problem from last week. Show that the length 4 subschemes
$$\operatorname{Spec}\mathbb{R}[x,y]/(x^2,y^2), \qquad \operatorname{Spec}\mathbb{R}[x,y]/(x^2-y^2,xy)$$
of $\operatorname{Spec}\mathbb{R}[x,y]$ are not isomorphic, but their fibre products with $\operatorname{Spec}\mathbb{C}$ over $\operatorname{Spec}\mathbb{R}$ are isomorphic. (Hint: one way to show that the original schemes are not isomorphic is to compute the solutions of the equation $z^2 = 0$ in both rings.)

2. Hartshorne II.3.7.

3. (Required) Hartshorne II.3.8.

4. Hartshorne II.3.21 (you should read II.3.20 first, but don't submit it).

5. (Required) Let X be a scheme, and let $\{U_i\}$ be an open affine cover of X.
   (a) Prove that X is separated if and only if for each pair of indices i,j, $U_i \cap U_j$ is affine and $\mathcal{O}(U_i \cap U_j)$ is generated (as a ring) by the images of $\mathcal{O}(U_i)$ and $\mathcal{O}(U_j)$. (You did part of this on the previous homework.)
   (b) Use this criterion to give another proof that $\mathbb{P}^n_{\mathbb{Z}}$ is separated.

6. (Difficult; definitely not required) Hartshorne II.4.10.

7. (Required) Describe the blowup of $\operatorname{Spec}\mathbb{Z}[x]$ at the point $(x,2)$, by writing down a cover by open affines and the glueing isomorphisms. Also describe the fibre over the point $(x,2)$.

8. Eisenbud-Harris III-25.

9. (Required if you have access to Eisenbud-Harris) Eisenbud-Harris III-43.

10. Eisenbud-Harris III-49.

11. (Required) Hartshorne II.5.11 and Hartshorne II.5.12 (these count as one exercise).

12. Hartshorne II.5.13.

13. Hartshorne II.5.14.

14. (Required) Hartshorne II.5.18.

15. (Required) Hartshorne II.7.1.
16. Hartshorne II.7.11(a) and (b).
17. Hartshorne II.7.12.