18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 6 (due Friday, March 20, in class)

Please submit exactly *eleven* of the following exercises, including all exercises marked "Required". Note that the Hartshorne exercises are now coming from both chapters II and III; read closely!

1. Eisenbud-Harris II-25 and II-26.

2. Eisenbud-Harris II-31.

3. (Required) Let $A \to B$ be a homomorphism of rings. Prove that B is faithfully flat over A in the sense of commutative algebra (i.e., for any A-module M, the adjunction map $M \to M \otimes_A B$ of A-modules is injective) if and only if $A \to B$ is flat and $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective (i.e., the morphism is faithfully flat in the sense of schemes).

4. (Required) Prove faithful flat descent for quasicoherent sheaves, as follows.

   (a) Let $f : A \to B$ be a faithfully flat ring homomorphism. Prove descent for the morphism $\operatorname{Spec}(B \otimes_A B) \to \operatorname{Spec} B$ induced by $f \otimes \mathrm{id}_B$, as follows. Let $\mu : B \otimes_A B \to B$ denote the multiplication map. Given a $(B \otimes_A B)$-module M equipped with a descent datum, recover the underlying B-module by forming $M \otimes_\mu B$.

   (b) Prove descent for any morphism between affine schemes, as follows. Let M be a B-module M equipped with a descent datum. Let $\pi_1, \pi_2 : B \to B \otimes_A B$ be the two embeddings. Let $\psi : M \otimes_{\pi_1} (B \otimes_A B) \to M \otimes_{\pi_2} (B \otimes_A B)$ be the descent isomorphism. Construct the A-submodule N of M consisting of those n for which $\psi(m \otimes 1) = m \otimes 1$. Show that the map $N \otimes_A B \to M$ is an isomorphism. (Hint: since $A \to B$ is faithfully flat, we may replace A and B by B and $B \otimes_A B$, respectively. Then use (a).)

   (c) Prove descent for any morphism of schemes, by reducing to the affine case.

5. Deduce Galois descent from faithful flat descent for quasicoherent sheaves, without using Theorem 90.

6. Prove that the following properties of a morphism of schemes can be checked after a faithfully flat quasicompact base change.

   (a) Universally closed.

   (b) Separated.

   (c) Proper.

7. A morphism is *quasifinite* if it is of finite type and has finite fibres (i.e., the inverse image of each point consists of finitely many points.)

(a) Prove that the property of being quasifinite is stable under base change.

(b) Prove that a morphism is finite if and only if it is proper and quasifinite.

(c) Prove that the properties of being quasifinite or finite can be checked after a faithfully flat quasicompact base change. (You may use the previous exercise whether or not you submit it.)

8. Let $f : X \to S$ be a morphism of schemes, and put $\Omega^i_{X/S} = \wedge^i_{\mathcal{O}_X} \Omega_{X/S}$ (the exterior product). Show that for each $i \geq 0$, there is a unique $f^{-1}(\mathcal{O}_S)$-linear homomorphism $d : \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ satisfying

$$d(g\, dh_1 \wedge \cdots \wedge dh_i) = dg \wedge dh_1 \wedge \cdots \wedge dh_i$$

for any open affine $U \subseteq X$ and any $g, h_1, \dots, h_i \in \mathcal{O}(U)$. (The hard part is well-definedness.) Show moreover that

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} \cdots$$

is a complex of $f^{-1}(\mathcal{O}_S)$-modules (the terms are actually $\mathcal{O}_X$-modules but the maps are not $\mathcal{O}_X$-linear). This complex is called the *de Rham complex* of X relative to S.

9. Here is a simple example of computing the cohomology of a de Rham complex. Let k be a field of characteristic zero. Let $P(x)$ be a polynomial with no repeated roots, and put $A = k[x, y]/(y^2 - P(x))$. Show that the cokernel of the map $d : A \to \Omega_{A/k}$ is finite-dimensional, and exhibit a basis. (You may wish to separate into cases depending on the parity of $\deg(P)$.)

10. Hartshorne II.8.5.

11. (Recommended) Hartshorne II.8.6.

12. Hartshorne II.8.8.

13. Prove directly (i.e., without using the characterization of smoothness in terms of flatness and regularity) that the morphism $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is formally smooth.

14. Hartshorne III.9.2.

15. (Required) Hartshorne III.9.3.

16. Hartshorne III.10.1.

17. (Required) Hartshorne III.10.6.