18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 7 (due Friday, April 3, in class)

Please submit exactly *nine* of the following exercises, including all exercises marked "Required". Besides any results from Hartshorne chapters I and II, you may also use the Riemann-Roch theorem (Theorem IV.1.3) and the Riemann-Hurwitz formula (Proposition IV.2.3 and its corollary).

1. (For those desiring more practice with blowing up) Eisenbud-Harris IV-45.

2. (Required) Hartshorne II.6.1.

3. Hartshorne II.6.4.

4. Hartshorne II.6.6.

5. Hartshorne IV.1.6.

6. Hartshorne IV.1.7.

7. (Required) Hartshorne IV.2.5.

8. Assume that k has characteristic different from 3. Prove that the group of automorphisms of the curve
$$x^3y + y^3z + z^3x = 0$$
in $\mathbb{P}^2_k$ contains a subgroup of order 168; since this curve has genus 3, this realizes the Hurwitz bound $84(g-1)$ in characteristic zero from the previous exercise. (The group of order 168 is in fact the full group of automorphisms even if the characteristic is nonzero, but you don't have to prove this.)

9. Hartshorne IV.3.1.

10. Hartshorne IV.5.2.

11. (Required) Let C be a nonhyperelliptic curve, and let $j : C \to \mathbb{P}^n_k$ denote the canonical embedding.

   (a) Prove that the image of j is not contained in any hyperplane.

   (b) Prove that the intersection of the image of j with any hyperplane has the same length, and compute that length. (This computes the *degree* of the embedding.)

12. Let C be a projective, irreducible, nonsingular curve over an algebraically closed field k. Suppose that $g(C) = 3$. Prove that either C is hyperelliptic, or C is isomorphic to a nonsingular curve in $\mathbb{P}^2_k$ of degree 4. (Hint: show that the canonical embedding is a closed immersion if and only if C is not hyperelliptic.)

13. Assume the field k has characteristic different from 2. Let $P(x) \in k[x]$ be a polynomial with no repeated roots, and let C be the normalization of the Zariski closure in $\mathbb{P}_k^2$ of the affine curve $\operatorname{Spec} k[x, y]/(y^2 - P(x))$.

   (a) Compute the genus of C by explicitly computing a basis of $H^0(X, \Omega_{X/k})$. (Hint: you might want to first change coordinates so that P has *odd* degree $2g + 1$. In that case, the genus should come out to equal g.)

   (b) Verify that $\deg(K) = 2g - 2$ by counting the zeroes of some nonzero section of $\Omega_{X/k}$.