18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 8 (due Friday, April 10, in class)

Please submit *nine* of the following exercises, including all items marked "Required". You may assume that all abelian categories under consideration admit a faithful additive functor to $\underline{\mathrm{Ab}}$ commuting with limits and colimits (so that diagram-chasing arguments become valid).

1. (Required) Let $\mathcal{C}$ be an abelian category. Prove that the category of complexes with values in $\mathcal{C}$ is again an abelian category.

2. (Required) Prove that an abelian group is injective if and only if it is divisible. (This proof was sketched in class.)

3. (Required) Let $T^\cdot : \mathcal{C}_1 \to \mathcal{C}_2$ be a cohomological functor between abelian categories, such that T^i is effaceable for each $i > 0$. Complete the proof that T is universal. (Hint: to check independence from the choice of u, you may want to use a pushout construction.)

4. Prove the acyclic resolution theorem. (Hint: break the acyclic resolution up into a sequence of short exact sequences, and take the long exact homology sequence of each piece.)

5. Prove that an element of $\underline{\mathrm{Mod}}_R$ is projective if and only if it is a direct summand of a free module.

6. It was stated in class that "exact functors preserve cohomology of complexes". Write down what this means formally and then prove it.

7. (Required) Suppose that the abelian category $\mathcal{C}$ admits enough injectives.
   (a) Prove that any complex in nonnegative degrees admits an injective resolution. (This is a bit ambiguous: here I mean a single complex of injectives receiving a map from the original complex which is a quasi-isomorphism.)
   (b) Prove that given any morphism $f : C^\cdot \to D^\cdot$ of complexes in nonnegative degrees, and any injective resolution $I^\cdot$ of $C^\cdot$, there exist an injective resolution $J^\cdot$ of $D^\cdot$ and a morphism $I^\cdot \to J^\cdot$ inducing f on cohomology.

8. (Required) Suppose that the abelian category $\mathcal{C}$ admits enough injectives. Write down a list of all of the compatibilities one must check in order to define right derived functors of a left exact functor F in terms of injective resolutions, but *do not* check them (except for the ones handled by the previous exercise). For instance, one of these is that the object $R^iF(X)$ is well-defined up to canonical isomorphism.

9. Let $\mathcal{C}$ be an abelian category.

(a) For objects M, N in $\mathcal{C}$ and $i > 0$, consider the set of equivalence classes of short exact sequences $0 \to M \to P \to N \to 0$ under the relation that two sequences with center terms P_1, P_2 are equivalent if there is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_1 & \longrightarrow & N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_M} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_N} & & \\ 0 & \longrightarrow & M & \longrightarrow & P_2 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

(this forces $P_1 \cong P_2$ by the five lemma, so this is indeed an equivalence relation). Prove that the operation of *Baer sum* gives this set a group structure: given two sequences with center terms P_1, P_2, let P be the quotient of $P_1 \times_N P_2$ by the image of the map $B \to P_1 \times_N P_2$ acting as $B \to P_1$ on the first factor and *minus* $B \to P_2$ on the second factor. (This is sometimes called the *Yoneda extension group*.)

(b) Prove that the construction in (a) is canonically isomorphic to the group $\mathrm{Ext}^1(N, M)$.

(c) In Ab, the short exact sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to P \to \mathbb{Z}/p\mathbb{Z} \to 0$ can be filled in in two ways, with $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$. However, $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Why are these two statements not contradictory?

10. Generalize the previous exercise as follows. For objects M, N in $\mathcal{C}$, consider classes of exact sequences $0 \to M \to P_n \to \cdots \to P_1 \to N \to 0$. Construct the minimal equivalence relation under which two sequences are equivalent if there is any commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_M} & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_N} & & \\ 0 & \longrightarrow & M & \longrightarrow & P'_n & \longrightarrow & \cdots & \longrightarrow & P'_1 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

but without any hypothesis on the arrows $P_i \to P'_i$. (Note that you need to *generate* the equivalence relation in this case; this condition itself is not symmetric.) Define Baer sum in this case to be

$$0 \to P''_n \to P_{n-1} \oplus P'_{n-1} \to \cdots \to P_2 \oplus P'_2 \to P''_1 \to 0$$

where P''_n is the fibred coproduct (pushout) of $M \to P_n$ and $M \to P'_n$, and P''_1 is the fibred product (pullback) of $P_1 \to N$ and $P'_1 \to N$.

(a) Prove that this construction gives a group. (This includes checking that the Baer sum respects equivalence.)

(b) Prove that this group is canonically isomorphic to $\mathrm{Ext}^i(M, N)$. (Hint: one way to do this is to check that these groups together form an effaceable cohomological functor. This means that given a short exact sequence $0 \to N_1 \to N \to N_2 \to 0$, you must interpret the connecting homomorphism $\delta^i : \mathrm{Ext}^i(M, N_2) \to \mathrm{Ext}^{i+1}(M, N_1)$ in terms of sequences.)

11. Let G be a group. Prove that for any left $\mathbb{Z}[G]$-module M, the first group cohomology group $H^1(G, M)$ is equal to the quotient of the group of *crossed homomorphisms* (i.e., set maps $f : G \to M$ satisfying $f(gh) = g(f(h)) + f(g)$ for all $g, h \in G$) by the subgroup of *principal crossed homomorphisms* (i.e., crossed homomorphisms of the form $f(g) = g(m) - m$ for some $m \in M$). (You may do this either using Yoneda extensions, or using an explicit construction of a projective resolution of $\mathbb{Z}$ in $\underline{\text{Mod}}_{\mathbb{Z}[G]}$. Beware that the Wikipedia explanation of the latter appears to have some typos.)

12. Prove that $\underline{\text{Mod}}_R$ has enough injectives as follows. Given $M \in \underline{\text{Mod}}_R$, let $M \to Q$ be a monomorphism of abelian groups with Q divisible. Prove that

$$M = \text{Hom}_R(R, M) \to \text{Hom}_{\underline{\text{Ab}}}(R, M) \to \text{Hom}_{\underline{\text{Ab}}}(R, Q)$$

may be viewed as a monomorphism of R-modules taking M into an injective R-module. (We will prove a more general result later.)

13. Let R be a ring and let M be an R-module. Prove that M is flat if and only if $\text{Tor}_1(R/I, M) = 0$ for every *finitely generated* ideal I of R. (Easy corollary: if R is a principal ideal domain, then M is flat if and only if it is torsion-free.)