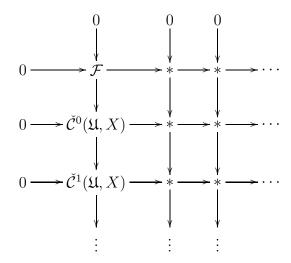
18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009) Problem Set 9 (due Friday, April 17, in class)

Please submit *nine* of the following exercises, including all items marked "Required".

1. (Required) Suppose that the (not necessarily paracompact) topological space X admits a nice basis B (i.e., a neighborhood basis closed under pairwise intersections). Let \mathcal{F} be a sheaf of abelian groups on X such that $\check{H}^i(U,\mathcal{F}) = 0$ for all $U \in B$. Prove that $\check{H}^i(X,\mathcal{F})$ is naturally isomorphic to $H^i(X,\mathcal{F})$ for all $i \geq 0$. This suffices for the proof that quasicoherent sheaves on affine schemes are acyclic, and that Čech cohomology computes the sheaf cohomology of any quasicoherent sheaf on a separated scheme (by taking B to be all affine subschemes). I'll address the comparison between Čech and sheaf cohomology for a paracompact topological space on the next problem set.

Hint: Induct on i. For the induction step, it suffices to prove that $\check{H}^i(\mathfrak{U}, \mathcal{F}) = H^i(X, \mathcal{F})$ for each cover \mathfrak{U} of X by basic opens. To do this, build a diagram



with exact rows and columns, with each * flasque. Then take global sections and argue using a diagram chase. (Again, this is secretly a spectral sequence argument!) See also Hartshorne Lemma III.4.4 for the definition of the natural map, and Hartshorne exercise III.4.4(c) for an argument in the case i=1.

2. (Required)

- (a) If you did not submit Hartshorne II.1.16, do it now.
- (b) Otherwise, answer the following question. Suppose I have a topological space X, a basis B, and a sheaf \mathcal{F} whose restriction map to each basic open set is surjective. Suppose also that

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of sheaves of abelian groups on X. Where does the proof that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective break down?

- 3. (Required) Hartshorne III.2.1(a).
- 4. Hartshorne III.2.1(b).
- 5. Hartshorne III.2.2.
- 6. (counts as two) Harsthorne III.2.3.
- 7. Hartshorne III.2.7.
- 8. (a) Do Hartshorne III.3.1 following the hint.
 - (b) Do Hartshorne III.3.1 again without using cohomology. (For this argument, you shouldn't need any noetherian hypotheses.)
 - (c) Do Hartshorne III.3.2, preferably without a noetherian hypothesis.
- 9. Hartshorne III.3.8.
- 10. Do Hartshorne III.4.1 but without the noetherian hypothesis.
- 11. (Required) Hartshorne III.4.3.
- 12. Hartshorne III.4.5.
- 13. Hartshorne III.4.7.
- 14. Using a suitable cover, compute the singular cohomology of the n-sphere S^n . You may assume without proof that the sheaf cohomology of $\underline{\mathbb{Z}}$ computes singular cohomology, and that this sheaf is acyclic on a contractible space (i.e., the higher singular cohomology of a contractible space vanishes). (Hint: your answer should be \mathbb{Z} in degrees 0, n and 0 otherwise. You may prefer to consider using an open cover in which each finite intersection is a disjoint union of contractible sets, but not necessarily having at most one connected component.)