18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Problem Set 9 (due Friday, April 17, in class)

Please submit *nine* of the following exercises, including all items marked "Required".

1. (Required) Suppose that the (not necessarily paracompact) topological space X admits a nice basis B (i.e., a neighborhood basis closed under pairwise intersections). Let $\mathcal{F}$ be a sheaf of abelian groups on X such that $\check{H}^i(U, \mathcal{F}) = 0$ for all $U \in B$. Prove that $\check{H}^i(X, \mathcal{F})$ is naturally isomorphic to $H^i(X, \mathcal{F})$ for all $i \geq 0$. This suffices for the proof that quasicoherent sheaves on affine schemes are acyclic, and that Čech cohomology computes the sheaf cohomology of any quasicoherent sheaf on a *separated* scheme (by taking B to be all affine subschemes). I'll address the comparison between Čech and sheaf cohomology for a paracompact topological space on the next problem set.

   Hint: Induct on i. For the induction step, it suffices to prove that $\check{H}^i(\mathfrak{U}, \mathcal{F}) = H^i(X, \mathcal{F})$ for each cover $\mathfrak{U}$ of X by basic opens. To do this, build a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathcal{C}}^0(\mathfrak{U}, X) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathcal{C}}^1(\mathfrak{U}, X) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

   with exact rows and columns, with each $*$ flasque. Then take global sections and argue using a diagram chase. (Again, this is secretly a spectral sequence argument!) See also Hartshorne Lemma III.4.4 for the definition of the natural map, and Hartshorne exercise III.4.4(c) for an argument in the case $i = 1$.

2. (Required)

   (a) If you did not submit Hartshorne II.1.16, do it now.

   (b) Otherwise, answer the following question. Suppose I have a topological space X, a basis B, and a sheaf $\mathcal{F}$ whose restriction map to each basic open set is surjective. Suppose also that
   $$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$
   is a short exact sequence of sheaves of abelian groups on X. Where does the proof that $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective break down?

3. (Required) Hartshorne III.2.1(a).
4. Hartshorne III.2.1(b).
5. Hartshorne III.2.2.
6. (counts as two) Harsthorne III.2.3.
7. Hartshorne III.2.7.
8. (a) Do Hartshorne III.3.1 following the hint.
   (b) Do Hartshorne III.3.1 again without using cohomology. (For this argument, you shouldn't need any noetherian hypotheses.)
   (c) Do Hartshorne III.3.2, preferably without a noetherian hypothesis.
9. Hartshorne III.3.8.
10. Do Hartshorne III.4.1 but without the noetherian hypothesis.
11. (Required) Hartshorne III.4.3.
12. Hartshorne III.4.5.
13. Hartshorne III.4.7.
14. Using a suitable cover, compute the singular cohomology of the n-sphere S^n. You may assume without proof that the sheaf cohomology of $\underline{\mathbb{Z}}$ computes singular cohomology, and that this sheaf is acyclic on a contractible space (i.e., the higher singular cohomology of a contractible space vanishes). (Hint: your answer should be $\mathbb{Z}$ in degrees $0, n$ and 0 otherwise. You may prefer to consider using an open cover in which each finite intersection is a *disjoint union* of contractible sets, but not necessarily having at most one connected component.)