In this lecture, we extend Serre duality to Cohen-Macaulay schemes over a field. As in the previous lecture, let \( k \) be a field (not necessarily algebraically closed), let \( j : X \to P = \mathbb{P}^N_k \) be a closed immersion with \( X \) of dimension \( n \), and let \( \mathcal{O}_X(1) \) be the corresponding twisting sheaf.

1 Cohen-Macaulay schemes and duality

Let \( \omega_X^2 \) denote a dualizing sheaf on \( X \); remember that this choice includes a trace map \( H^n(X, \omega_X^2) \to k \). We then obtain natural functorial maps

\[
\theta^i : \text{Ext}^i_X(\mathcal{F}, \omega_X^2) \to H^{n-i}(X, \mathcal{F})
\]

because both sides are cohomological functors on the opposite category of coherent sheaves on \( X \), and the one on the left is effaceable because it vanishes on direct sums of twisting sheaves. By the definition of a dualizing sheaf, \( \theta^0 \) is always an isomorphism.

**Theorem.** The following conditions are equivalent.

(a) The scheme \( X \) is equidimensional (each irreducible component has dimension \( n \)) and Cohen-Macaulay.

(b) The maps \( \theta^i \) are isomorphisms for all \( i \geq 0 \) and all coherent sheaves \( \mathcal{F} \) on \( X \).

This is of course meaningless if I don’t tell you what a Cohen-Macaulay scheme is. For the moment, suffice to say that a scheme is Cohen-Macaulay if and only if each of its local rings is a Cohen-Macaulay ring. That already has content, because then the theorem says that (b) is equivalent to a local condition on \( X \), which is far from obvious.

I’ll also point out that a regular local ring is always Cohen-Macaulay. This implies the following.

**Corollary.** If \( X \) is smooth over \( k \), then \( \theta^i \) is an isomorphism for all \( i \geq 0 \) and all coherent sheaves \( \mathcal{F} \) on \( X \).

2 Proof of the duality theorem, part 1

Even without knowing what a Cohen-Macaulay scheme is, we can at least start working to prove that condition (b) is equivalent to a local condition on \( X \). Let us start by relating (b) to two global vanishing assertions.

**Lemma.** The following conditions are equivalent to (b).
(c) For any locally free coherent sheaf $\mathcal{F}$ on $X$, for $q$ sufficiently large, we have $H^i(X, \mathcal{F}(-q)) = 0$ for all $i < n$.

(c') For $q$ sufficiently large, we have $H^i(X, \mathcal{O}_X(-q)) = 0$ for all $i < n$.

Note that condition (c) is a sort of opposite to Serre’s vanishing theorem, which gives the vanishing of $H^i(X, \mathcal{F}(q))$ for $i > 0$ and $q$ sufficiently large.

**Proof.** Given (b), for any locally free coherent sheaf $\mathcal{F}$ on $X$, we have

$$H^i(X, \mathcal{F}(-q)) = \text{Ext}^{n-i}_X(\mathcal{F}(-q), \mathcal{O}_X)$$

$$= \text{Ext}^{n-i}_X(\mathcal{O}_X, \mathcal{F}^\vee \otimes \mathcal{O}_X(q))^\vee$$

$$= H^{n-i}(X, \mathcal{F}^\vee \otimes \mathcal{O}_X(q))^\vee$$

and this vanishes for $n - i > 0$ and $q$ large by Serre’s vanishing theorem. Thus (b) implies (c).

It is clear that (c) implies (c'). Given (c'), it follows that $H^{n-i}(X, \cdot)^\vee$ is effaceable for all $i > 0$ since we can cover $\mathcal{F}$ with a direct sum of twisting sheaves. Hence $\theta^i$ is the natural map between two universal cohomological functors, hence is an isomorphism. Thus (c') implies (b).

We next reformulate this in local terms, using Serre duality on $P$.

**Lemma.** The following condition is equivalent to (b).

(d) For all $i < n$, $\text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P) = 0$.

Remember that no matter what $X$ is, we have $\text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P) = 0$ for $i > n$: we proved this in the course of constructing the dualizing sheaf $\omega_X^\vee$.

**Proof.** By Serre duality on $P$ (and choosing an isomorphism $H^n(P, \omega_P) \cong k$), we may identify

$$H^i(X, \mathcal{O}_X(-q)) \cong H^i(P, j_* \mathcal{O}_X(-q)) \cong \text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P(q))^\vee.$$

So (c) is equivalent to the condition that for $q$ sufficiently large, $\text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P(q)) = 0$ for all $i < n$. Recall from earlier that for $q$ large,

$$\text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P(q)) = \Gamma(P, \text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P(q))) = \Gamma(P, \text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P)(q)).$$

Since $\text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P)$ is coherent, $\Gamma(P, \text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P)(q))$ vanishes for $q$ sufficiently large if and only if $\text{Ext}^n_P(j_* \mathcal{O}_X, \omega_P) = 0$.

Condition (d) can be rewritten as follows.

**Lemma.** The following condition is equivalent to (b).

(e) For each point $x \in X$, if $A = \mathcal{O}_{P,x}$ and $I$ is the ideal of $A$ defining $X$ at $x$, then for all $i < n$, $\text{Ext}^n_A(A/I, A) = 0$.

**Proof.** This translates directly from (d) once we remember that $\omega_P$ is locally free of rank 1 on $P$.

This is almost the local condition we are seeking, except that it still refers to the position of $X$ within $P$.
3 The Cohen-Macaulay condition

To get rid of the dependence of our duality condition on the relative geometry of $X$ within $P$, we need some more sophisticated commutative algebra.

**Proposition.** Let $A$ be a regular local ring and let $M$ be a finitely generated $A$-module. Then for any nonnegative integer $n$, the following are equivalent.

(a) We have $\text{Ext}^i(M, A) = 0$ for all $i > n$.

(b) For any $A$-module $N$, we have $\text{Ext}^i(M, N) = 0$ for all $i > n$.

(c) There exists a projective resolution $0 \to L_n \to \cdots \to L_1 \to L_0 \to M \to 0$ of $M$ at length at most $n$.

**Proof.** See Hartshorne Proposition III.6.10A (and associated Matsumura reference) and exercise III.6.6.

The smallest integer for which this holds is called the *projective dimension* of $M$ (if it exists), denoted $\text{pd}_A(M)$. For instance, $M$ is projective if and only if $\text{pd}_A(M) = 0$.

For $M$ a module over a ring $A$, a *regular sequence* is a sequence $x_1, \ldots, x_n$ of elements of $A$ such that for $i = 1, \ldots, n$, $x_i$ is not a zerodivisor on $M/(x_1, \ldots, x_{i-1})M$. For $A$ a local ring, the *depth* of $M$ is the maximal length of a regular sequence with all $x_i$ in the maximal ideal of $A$.

**Proposition.** For $A$ a regular local ring and $M$ an $A$-module,

$$\text{pd}_A(M) + \text{depth}_A(M) = \dim(A).$$


We can finally give a local equivalent to condition (b) from the duality theorem. Recall that our last equivalent (e) said that for each $x \in X$, for $A = O_{P,x}$ and $I$ the ideal of $A$ defining $X$ at $x$, $\text{Ext}_A^{N-i}(A/I, A) = 0$ for all $i < n$. This is equivalent to $\text{pd}_A(A/I) \leq N - n$, and hence to $\text{depth}_A(A/I) \geq n$. The trick is that if $M$ is an $A/I$-module, then $\text{depth}_A(M) = \text{depth}_{A/I}(M)$. Thus we have the following.

**Lemma.** The following condition is equivalent to (b).

(f) For each point $x \in X$, if $B = O_{X,x}$, then $\text{depth}_B(B) \geq n$.

On the other hand, we always have $\text{depth}_B(B) \leq \dim(B) \leq n$, so it is equivalent to require $\text{depth}_B(B) = \dim(B) = n$.

This condition $\text{depth}_B(B) = \dim(B)$ is in fact the definition of a *Cohen-Macaulay* local ring $B$. Any regular local ring is Cohen-Macaulay, since we can use generators of the cotangent space as a regular sequence. But the Cohen-Macaulay condition is much more permissive; for instance, any *local complete intersection* is Cohen-Macaulay.