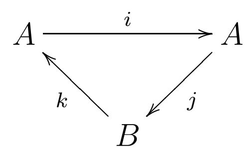
18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Spectral sequences and Čech cohomology

We explain the construction (or rather, one particular construction) of spectral sequences, enough to explain how they are used as part of the computation of the sheaf cohomology of quasicoherent sheaves on affine schemes using Čech cohomology.

I continue to recommend Bott and Tu, *Differential Forms in Algebraic Topology* as a good reference for spectral sequences.

1 Exact couples

It is handy to start with the following bit of homological algebra. An *exact couple* is a circular exact sequence

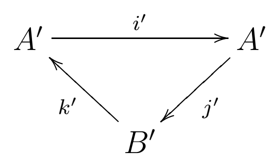


For instance, given an exact sequence $0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0$, we get an exact couple by taking $k = 0$. A more typical example: given an exact sequence of complexes

$$0 \to A^\cdot \to A^\cdot \to B^\cdot \to 0,$$

we get an exact couple involving the total cohomologies $\oplus_i h^i(A^\cdot)$ and $\oplus_i h^i(B^\cdot)$ using the long exact sequence in cohomology.

From an exact couple we obtain a *derived exact couple*



as follows.

- Define $d : B \to B$ as $d = j \circ k$. Then $d \circ d = j \circ k \circ j \circ k = 0$ because $k \circ j = 0$, so I can define the cohomology $B' = h(B) = \ker(d)/\operatorname{im}(d)$.
- Put $A' = \operatorname{im}(i)$.
- We now have an obvious map $i' : A' \to A'$ induced by i.
- We now claim that there is a well-defined map $j' : A' \to B'$ sending $i(a)$ to the class of $j(a)$ for any $a \in A$. To make sense of this, we first note that $j(a) \in \ker(d)$ because $j \circ k \circ j = 0$. We next note that if $i(a) = 0$, then $a = k(b)$ for some $b \in B$ by exactness, so $j(a) = k(j(b)) = d(b)$.

- We now claim there is a well-defined map $k' : B' \to A'$ induced by k. That is, if $b \in \ker(d)$, k' should carry the class of k' to $k(b)$; this belongs to $\mathrm{im}(a)$ because $(j \circ k)(b) = 0$, so $k(b) = i(a)$ for some $a \in A$ by exactness. This is well-defined:

It is a routine exercise in diagram chasing to verify that this is again exact.

2 Filtered complexes and double complexes

Let $C^\cdot$ be a cohomologically graded complex in nonnegative degrees. A *filtration* on $C^\cdot$ is a decreasing sequence of subcomplexes

$$C^\cdot = \mathrm{Fil}^0 C^\cdot \supseteq \mathrm{Fil}^1 C^\cdot \supseteq \cdots .$$

The associated *graded complex* is

$$\mathrm{Gr}^i C^\cdot = \mathrm{Fil}^i C^\cdot / \mathrm{Fil}^{i+1} C^\cdot.$$

For instance, suppose $D^{p,q}$ is a *double complex*, with differerentials $d_p^\cdot$ in the p-direction and $d_q^\cdot$ in the q-direction. We form a single complex

$$C^k = \bigoplus_{p+q=k} D^{p,q}$$

with derivation $d_p + (-1)^p d_q$. (The alternating sign is needed to ensure that this is actually a complex.) We then obtain a filtration on $C^\cdot$ by setting

$$\mathrm{Fil}^i C^k = \bigoplus_{p+q=k, p \geq i} D^{p,q}.$$

3 The spectral sequence of a filtered complex

Given a filtered complex $C^\cdot$, there are two interesting invariants one can consider. Perhaps the most natural one is the cohomology $h^\cdot(C^\cdot)$, equipped with the decreasing filtration

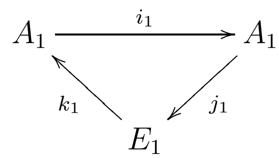
$$\mathrm{Fil}^i h^\cdot(C^\cdot) = \mathrm{im}(h^\cdot(\mathrm{Fil}^i C^\cdot)).$$

However, in practice this will usually be something complicated. A less complicated invariant will be the cohomology of the graded complex $h^\cdot(\mathrm{Gr}^p C^\cdot)$. This is a rather crude approximation to the cohomology of the total complex; it turns out that there is a sequence of refinements that give closer and closer approximations. These constitute the *spectral sequence* associated to the filtered complex.

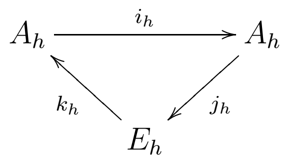
To start with, take the exact sequence of complexes

$$0 \to \bigoplus_{p \in \mathbb{Z}} \mathrm{Fil}^{p+1} C^\cdot \to \bigoplus_{p \in \mathbb{Z}} \mathrm{Fil}^p C^\cdot \to \bigoplus_{p \in \mathbb{Z}} \mathrm{Gr}^p C^\cdot \to 0.$$

Identifying the first two members by shifting indices, then taking the long exact sequence in cohomology, we get an exact couple



in which $E_1 = \bigoplus_{p \in \mathbb{Z}} h^{\cdot}(\mathrm{Gr}^p C^{\cdot})$. By repeatedly extracting derived exact couples, we get a sequence of exact couples



for $h = 1, 2, \ldots$. The *spectral sequence* here is specifically the sequence of groups E_h equipped with the square-zero endomorphisms $d_h = j_h \circ k_h$. Note that E_{h+1} is just the cohomology of E_h for d_h; the mysterious part is where the next map d_{h+1} comes from. (The terms in this sequence are often called the *sheets*, or *pages*, of the spectral sequence. The visual significance of these metaphors may become more clear in the next section.)

Without any additional hypotheses, the spectral sequence does not say much. But under certain circumstances, the E_h "converge" to something useful. Namely, suppose that the complex $C^{\cdot}$ comes not only with a filtration but with a grading $C^{\cdot} = \oplus_q C^{\cdot}_q$.

Theorem. *Suppose that for each q, the induced filtration on $C^{\cdot}_q$ has only finitely many distinct steps. Then the spectral sequence* converges*, in the sense that for each q, the q-th graded piece of E_h stabilizes for h large. If we let E_∞ denote the sum of the stable graded pieces, then E_∞ is canonically isomorphic to the associated graded group of the filtered cohomology* $\mathrm{Fil}^i\, h^{\cdot}(C^{\cdot})$.

Note that we still don't quite manage to compute the filtered cohomology, but but only its graded pieces. Still, that information itself is often very very useful. (It is sometimes said that the spectral sequence *abuts* to the filtered cohomology.)

Proof. See Bott and Tu, Theorem 14.6. □

4 The spectral sequence of a double complex

Let us see how this works in the specific example of a double complex. (I'm just going to state the result; see Bott and Tu for the derivation.) Let $D^{p,q}$ be a double complex, and let $C^{\cdot}$ be the associated filtered single complex. It is customary to draw pictures in this orientation:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
D^{0,2} & D^{1,2} & D^{2,2} & \cdots \\
D^{0,1} & D^{1,1} & D^{2,1} & \cdots \\
D^{0,0} & D^{1,0} & D^{2,0} & \cdots
\end{array}$$

without any arrows (at least for now).

Let me redraw this picture writing $E_0^{p,q}$ for $D^{p,q}$, and drawing in the vertical arrows standing for $(-1)^p d_q$:

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\uparrow & \uparrow & \uparrow & \\
E_0^{0,2} & E_0^{1,2} & E_0^{2,2} & \cdots \\
\uparrow & \uparrow & \uparrow & \\
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} & \cdots \\
\uparrow & \uparrow & \uparrow & \\
E_0^{0,0} & E_0^{1,0} & E_0^{2,0} & \cdots
\end{array}$$

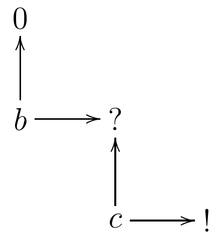
Taking cohomology here gives you exactly E_1. A quick diagram chase shows that the next differential is precisely the one induced by d_p:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} & \longrightarrow & \cdots \\
E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} & \longrightarrow & \cdots \\
E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & \cdots
\end{array}$$

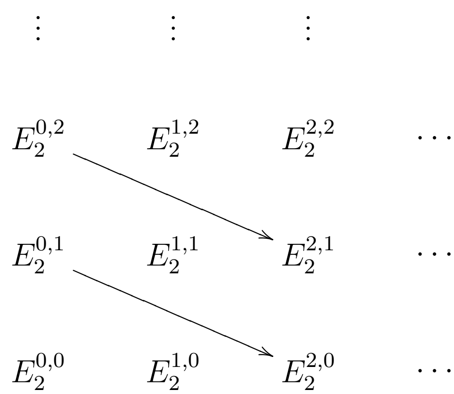
Taking cohomology gives the next sheet E_2. But what is the next differential? Again, I'll just state the answer. Each element of E_2 is represented by an element of b for which for some c,

$$d_q(b) = 0, \qquad d_p(b) = (-1)^{p+1} d_q(c).$$

The next differential carries this class to $d_p(c)$, which turns out to be well-defined.



That is, our next page should be drawn like this:



The pattern continues: we have

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and we can explicitly see the stabilization, since we get an increasingly large bottom left corner with no arrows to or from anyplace other than 0. Let $E_\infty^{p,q}$ denote the stable values; then the associated graded complex of the filtered total cohomology has k-th step given by

$$\bigoplus_{p+q=k} E_\infty^{p,q}.$$

5 Spectral sequences and Čech cohomology

Here is how spectral sequences make quick work of the comparison theorem between Čech and sheaf cohomology, in the form needed for algebraic geometry. Let X be a topological space, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{I}^\cdot$ be a flasque resolution of $\mathcal{F}$. Take the double complex

$$D^{p,q} = \check{C}^p(X, \mathcal{I}^q) = \varinjlim_{\mathfrak{U}} \check{C}^p(\mathfrak{U}, \mathcal{I}^q).$$

The trick here is that there are *two* different ways to run the spectral sequence construction from a double complex, depending on how you orient the diagram. As written, we first take Čech cohomology, and then take cohomology of whatever that yields:

$$E_{1a}^{p,q} = \check{H}^p(X, \mathcal{I}^q)$$
$$E_{2a}^{p,q} = h^q(\check{H}^p(X, \mathcal{I}^\cdot)).$$

Note that $E_{1a}^{p,q} = 0$ for $p > 0$ because the Čech cohomology of a flasque sheaf is zero, whereas $E_{1a}^{p,0} = \Gamma(X, \mathcal{F})$. Thus $E_{2a}^{p,q} = 0$ for $p > 0$, and in fact $E_{2a}^{p,q} = E_{\infty a}^{p,q}$ for all p, q. Since we only have one term along each antidiagonal, we actually get much more than usual: we really have computed the cohomology of the total complex, and it is the $E_{2a}^{0,q} = H^q(X, \mathcal{F})$.

Now let's run the spectral sequence with the roles of p and q reversed. This time, I take cohomology in the q-direction first, so I start with

$$E_{1b}^{q,p} = h^q(\check{C}^p(X, \mathcal{I}^\cdot)).$$

This is a rather strange object, but we can repackage it in a useful way by noting that the functor $\mathcal{I} \to \check{C}^p(X, \mathcal{I})$ preserves *exact sequences of presheaves*, i.e., sequences of presheaves where the sections over any open give an exact sequence. That means that working with presheaves, I can commute the cohomology computation across the $\check{C}^p$. I'll take advantage of this by defining the presheaf $\mathcal{H}^q$ by

$$\mathcal{H}^q(U) = H^q(\mathcal{I}^\cdot(U)) = H^q(U, \mathcal{F}),$$

so that

$$E_{1b}^{q,p} = \check{C}^p(X, \mathcal{H}^q)$$
$$E_{2b}^{q,p} = \check{H}^p(X, \mathcal{H}^q)$$

interpreted as the Čech complex associated to a presheaf (defined using the same formula as for sheaves). This spectral sequence must converge to some term $E_{\infty b}^{q,p}$ giving graded pieces of the total cohomology, which we already identified as the sheaf cohomology of $\mathcal{F}$ itself.

This isn't useful as an abstract method for dealing with Čech cohomology. However, it is just the thing I need to prove the theorem that I need to finish the argument that quasicoherent sheaves on affine schemes are acyclic.

Theorem. *Let X be a topological space equipped with a nice basis B (i.e., a basis closed under pairwise intersections; we need not assume $X \in B$). Let $\mathcal{F}$ be a sheaf of abelian groups on X such that $\check{H}^i(U, \mathcal{F}) = 0$ for all $i > 0$ and all $U \in B$. Then there are natural isomorphisms $\check{H}^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$ for all $i \geq 0$.*

Proof. The natural maps come from the fact that if $\mathcal{I}^\cdot$ is an injective resolution of $\mathcal{F}$, then the Čech resolution $\check{\mathcal{C}}^\cdot(X, \mathcal{F})$ admits a map into $\mathcal{I}^\cdot$ which is a quasi-isomorphism, and is well-determined up to a chain homotopy. (This is similar to the homework problem about injective resolutions of complexes; see PS 8, problem 7.)

To prove the theorem, it suffices to check for X equal to an open in B, as then the Leray theorem asserts that we can compute sheaf cohomology using any cover by elements of B, and any open cover refines to such. So assume hereafter $X \in B$.

We induct on i, the case $i = 0$ being an easy consequence of the sheaf axiom. Say we know that

$$H^j(U, \mathcal{F}) = 0 \qquad (0 < j < i, U \in B).$$

Then the spectral sequence $E_{\cdot b}$ from above has $E_{2b}^{q,p} = 0$ for $0 < q < i$. By staring at the spectral sequence, we see that the terms with $q + p = i$ must already be stable, so the total i-th cohomology must just be

$$E_{2b}^{0,i} = \check{H}^i(X, \mathcal{H}^0) = \check{H}^i(X, \mathcal{F}).$$

Since we also know that the total cohomology is $H^i(X, \mathcal{F})$, we obtain the desired isomorphism. □