More on coherent sheaves


Addendum: comment about generic fibres

I forgot to mention in the previous handout that the generic fibre construction loses information in a very precise sense. If \( P \) is a formal scheme of finite type over \( o_K \), and \( P' \) is a blowup of \( o_K \) within the special fibre, then the induced map from the generic fibre of \( P' \) to that of \( P \) is an isomorphism.

Example: if \( P = \text{Spf} \ o_K\langle x, y \rangle \), and \( P' \) is the blowup along the ideal \((x, y, \pi)\) for some \( \pi \in m_K \), then in both cases the generic fibre consists of the closed points of the affine plane over \( K \) represented by geometric points \((a, b)\) with \(|a| \leq 1 \) and \(|b| \leq 1 \).

In fact, Raynaud proved an equivalence of categories between a certain category of rigid spaces, and a certain category of formal schemes localized at the blowups in the special fibre. (I don’t know the precise formulation offhand; will go look it up later.) This cuts both ways: sometimes it’s easier to work with rigid spaces than formal schemes because you would just as soon ignore the blowups in the special fibre. But sometimes you can’t make a construction in the rigid setting without keeping track of some choice of an associated formal scheme (as in Laurent Fargues’s seminar talk a couple of weeks ago).

Coherent sheaves on affinoid spaces

Last time I defined a coherent sheaf on a rigid space to be a sheaf which, locally on some admissible affinoid covering, is isomorphic to the sheaf generated by a finitely generated module over the corresponding affinoid algebra. The Gerritzen-Grauert-Tate theorem implies that you actually get a sheaf this way, but it doesn’t imply that this sheaf has “enough” sections. This is fixed by the following theorem of Kiehl, whose proof I’ll sketch here; see [FvdP, Section 4.5] for details.

**Theorem 1 (Kiehl).** Let \( \mathcal{F} \) be a coherent sheaf on an affinoid space \( X = \text{Max}(A) \), and put \( M = \mathcal{F}(X) \). Then \( M \) is finitely generated, and \( \mathcal{F} \) is isomorphic to the coherent sheaf associated to \( M \).

**Sketch of proof.** As in the Gerritzen-Grauert-Tate argument, one reduces to the case of a Laurent covering given by a single \( f \). (See [FvdP, Section 4.5] for more on this reduction.) That puts us in the following situation. We are given \( f \in A \), finitely generated modules \( M_+ \) and \( M_- \) over \( A(f) \) and \( A(f^{-1}) \), respectively, and an isomorphism between \( M_+ \otimes A(f, f^{-1}) \)
and $M_\perp \otimes A\langle f, f^{-1}\rangle$. For $M = \mathcal{F}(X)$, we must show that the maps $M \otimes A\langle f \rangle \to M_+$ and $M_+ \otimes A\langle f^{-1} \rangle \to M_-$ are surjective.

One can easily reduce this to just checking for $M_-$ (as in [FvdP, Lemma 4.5.5], but they seem to get it backwards; see below). For this part, one needs a form of Cartan’s lemma: there exists $c > 0$ such that any invertible $n \times n$ matrix $U$ over $A\langle f, f^{-1} \rangle$ with $\|U - I_n\| < c$ (where the matrix norm is the maximum over entries) factors as $U_+U_-$, with $U_+$ invertible over $A\langle f \rangle$ and $B$ invertible over $A\langle f^{-1} \rangle$. See [FvdP, Lemma 4.5.3] for this calculation. (The basic idea is to split $U$ additively as $I + V_+ + V_-$, where $V_+$ has only nonnegative powers of $V$, $V_-$ has only nonnegative powers of $V_-$, and both $V_+$ and $V_-$ have small norm. Then you replace $U$ by $(1 - V_+)(1 - V_-)$ and repeat; if $c$ is small enough, this process converges to the desired factorization.)

To check surjectivity of $M \otimes A\langle f \rangle \to M_+$ now, you choose a set of $n$ generators of $M_+$ and a set of $n$ generators of $M_-$ (for some $n$), and write down change-of-basis matrices $U$ and $V$ between the two sets of generators over $A\langle f, f^{-1} \rangle$. Since the image of $A$ in $A\langle f \rangle$ is dense, we can approximate $U$ closely by a matrix over $A$, so that $\|(U' - U)V\| < c$ with $c$ as above. We can then factor $I_n - (U' - U)V = U_+U_-$ as above, and changing basis from $M_+$ via $U_+$ gives us a set of generators which are defined on both subspaces, yielding basis.

Given the surjectivity, we can choose a finitely generated submodule $M_1$ of $M$ which generates both $M_+$ and $M_-$. Let $\mathcal{M}_1$ be the associated sheaf: then $\mathcal{M}_1 \to \mathcal{F}$ is surjective. Let $\mathcal{G}$ be the kernel of that map; then $\mathcal{G}$ is also coherent and given by finitely generated modules on the two pieces of the cover, so we can find a surjection $\mathcal{M}_2 \to \mathcal{G}$, where $\mathcal{M}_2$ is the coherent sheaf associated to the finitely generated module $M_2$. Hence $\mathcal{F}$ is the cokernel of the map $\mathcal{M}_2 \to \mathcal{M}_1$; by acyclicity, its global sections are precisely $M_2/M_1$, and the associated sheaf is precisely $\mathcal{F}$ because they match up on the two opens.

This means that we can glue coherent sheaves on any admissible cover of $X$ (I may have said this earlier without justification).

**Example: the sheaf of differentials**

An important example of a coherent sheaf is the sheaf of continuous differentials.

**Theorem 2.** Let $A$ be an affinoid algebra. There exists a finitely generated $A$-module $\Omega_{A/K}$ equipped with a $K$-linear derivation $d : A \to \Omega_{A/K}$ with the following universal property: given any finitely generated $A$-module $M$ equipped with a $K$-linear derivation $D : A \to M$, there exists a unique $A$-module homomorphism $l : \Omega_{A/K} \to M$ with $D = l \circ d$.

**Proof.** In case $A = T_n = K\langle x_1, \ldots, x_n \rangle$, this is easy to check: take $\Omega_{A/K} = A dx_1 + \cdots + A dx_n$ together with the formal total differential. That is,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i,$$
where the partial derivation is done formally on the series.

For general $A$, pick a presentation $A \cong T_n/(f_1, \ldots, f_m)$, and put

$$\Omega_{A/K} = \Omega_{T_n/K}/(T_n df_1 + \cdots + T_n df_m),$$

with the induced derivation from $T_n$. \hfill \square

The module $\Omega_{A/K}$ is called the **module of finite differentials**, or the **universal finite differential module**. It is obviously unique up to unique isomorphism, so one gets from it a coherent sheaf of differentials $\Omega_X$ on any rigid space $X$.

Note that the module of finite differentials is not always the same as the module of Kähler differentials, because the latter is sometimes badly behaved. In fact, it can be shown (though I don’t remember a reference offhand) that the module of finite differentials is the maximal separated quotient of the completion of the ordinary module of Kähler differentials; it is thus sometimes called the **module of continuous differentials**, because the latter description makes it universal for continuous derivations into any Banach $A$-algebra.

Here are some fun facts about modules of differentials. (See [FvdP, Theorem 3.6.3], although they mostly just refer to Springer LNM 38 “Differentialrechnung in der analytische Geometrie”):

**Proposition 3.** Let $A$ be an affinoid algebra which is an integral domain, and let $F$ be its fraction field.

(a) The dimension of $\Omega_{A/K} \otimes_A F$ over $F$ equals the Krull dimension $d$ of $A$.

(b) Let $m$ be a maximal ideal of $A$. Then the following are equivalent.

(i) $A$ is smooth over $K$ at $m$. (That means that locally near $m$, $A$ can be written as the vanishing locus in some affine space $\text{Max} T^n$ of some number $m$ of functions whose $m \times n$ matrix of partial derivatives has maximal rank. If $K$ is perfect, this is equivalent to $A$ being regular at $m$, i.e., the localization $A_m$ being a regular local ring.)

(ii) The localization $A_m \otimes_A \Omega_{A/K}$ is free over $A_m$ (of rank $d$).

(iii) The dimension of $\Omega_{A/K}/m\Omega_{A/K}$ over $A/m$ is $d$.

**Closed analytic subspaces**

Let $X$ be a rigid space over $k$, and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}$. Then we get a subspace of $X$ associated to $\mathcal{I}$ whose points are the support of the coherent sheaf $\mathcal{O}/\mathcal{I}$ (with topology induced from $X$), and whose structure sheaf is the restriction of $\mathcal{O}/\mathcal{I}$. On an affinoid space, this of course corresponds to viewing the vanishing locus of some ideal of functions as the Max of the quotient affinoid algebra. Any such subspace of $X$ is called a **closed analytic subspace**.
Next time: GAGA

Next time, I’ll talk about separatedness and properness for rigid spaces and sketch the proof of Kiehl’s rigid GAGA theorem.

Exercises

1. Assume that $K$ has characteristic zero. Let $X$ be a smooth rigid space, let $\mathcal{F}$ be a coherent sheaf on $X$, and suppose that there exists a $K$-linear connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/K}$. That is, given a section $s$ of $\mathcal{F}$ and a function $f$, we have $\nabla(sf) = f\nabla(s) + s \otimes df$. Prove that $\mathcal{F}$ must be locally free. (Hint: replace the local ring at a maximal ideal by its completion and check there that the torsion submodule must vanish.)