To keep things moving, I’m going to be terse now (as in [GH1]); there are lots of details filled in in [GH2].

**References:** Jay notes that [GH1] (the terse one) is available online; I’ll put a link on the notes page.

**Corrections from last time**

Thanks to Jay for these.

page 2: the first Lubin-Tate paper only deals with height 1 (that is all that is needed for local class field theory), and they show uniqueness only over the completion of the maximal unramified extension of $K$. (Indeed, the fact that they are not isomorphic is key to being able to use them for explicit local reciprocity!)

page 3, top of page: “free $R[[X]]$ over $x_1, \ldots, x_n$” should be over $dx_1, \ldots, dx_n$. By $\omega(F(X,Y))$, I mean you write $\omega(X) = f_1(X)dx_1 + \cdots + f_n dx_n$, then you write $F(X,Y) = (F_1(X,Y), \ldots, F_n(X,Y))$, you plug in

$$\omega(F(X,Y)) = f_1(F(X,Y))dF_1(X,Y) + \cdots + f_n(F(X,Y))dF_n(X,Y),$$

then expand each $dF_i(X,Y)$ by the chain rule.

page 3, Height: change “formal group” in the second line to “formal $\mathfrak{m}$-module” and scratch the reference to formal $\mathfrak{m}$-modules in the parenthetical.

**A group action**

Last time, we built a universal deformation $F$ over $A = \mathfrak{o}[v_1, \ldots, v_{h-1}]$ of a formal $\mathfrak{o}$-module over $\mathbb{F}_q$ of height $h$, which I’ll denote by $F_0$. That means that the group $G = \text{Aut}(F_0)$ acts on the deformation space $\text{Spf} A$, and on the corresponding rigid analytic space $X$, which is the open unit polydisc in $v_1, \ldots, v_{h-1}$.

It turns out that $D = \text{End}(F_0)$ is a division algebra of degree $n$, $G$ is the group of units in some maximal order therein, and $D \otimes K \cong M_n(K)$ is split. That means $G$ has a natural $n$-dimensional linear representation $V_K$ over $K$, as does $D^*$. In particular, $G$ acts on the hyperplanes of $V_K$, i.e., on $\mathbb{P}(V_K^\vee)$; the latter carries a rigid analytic space structure, and the group action is by analytic morphisms.

The crystalline period mapping, to be defined, is a rigid analytic $G$-equivariant étale morphism $\Phi : X \to \mathbb{P}(V_K^\vee)$ which classifies deformations “up to isogeny” as follows. For $A$ an affinoid algebra over $K$, let $F_a$ and $F_b$ be deformations of $F_0$ over $\mathfrak{a}_A$, corresponding to points $a, b \in X(A)$. Then an isogeny of $F_0$, viewed as an element $T \in D^*$, deforms to an isogeny $F_a \rightarrow F_b$ if and only if $T\Phi(a) = \Phi(b)$. 
The universal additive extension

In order to specify \( \Phi \), I have to give you a \( G \)-equivariant line bundle \( \mathcal{L} \) on \( X \) and a \( K[G] \)-homomorphism \( V_K \to H^0(X, \mathcal{L}) \) whose image is basepoint-free (i.e., the images don’t all vanish at a point). For \( x \in X \), we then take \( \Phi(x) \) to be the hyperplane of \( V_K \) which maps to sections of \( \mathcal{L} \) vanishing at \( x \).

First, \( \mathcal{L} \) is the inverse of the analytification of the sheaf \( \omega \) of invariant differentials, \( a/k/a \) the Lie algebra \( \text{Lie}(F) \). In order to make the map, I must consider the universal additive extension \( E \) of \( F \); it sits in an exact sequence

\[
0 \to N \to E \to F \to 0
\]

with \( N = \mathbb{G}_a \otimes \text{Ext}(F, \mathbb{G}_a)^{\vee} \), and it is universal: if \( 0 \to N' \to E' \to F \to 0 \) is an extension of \( F \) by an additive \( \alpha \)-module, then there are unique homomorphisms \( i : E \to E' \), \( j : N \to N' \) such that

\[
\begin{array}{ccc}
0 & \to & N \\
\downarrow j & & \downarrow i \\
N' & \to & E \\
\downarrow i & & \downarrow \text{id}_E \\
0 & \to & F \\
\end{array}
\]

commutes. (This is straightforward, modulo some cohomology arguments which have already been exploited in constructing the universal deformation, namely, that \( \text{Hom}(F, \mathbb{G}_a) = 0 \) and \( \text{Ext}(F, \mathbb{G}_a) \) is free of rank \( n - 1 \). See [GH2, Proposition 11.3].) On the level of Lie algebras, we have

\[
0 \to \text{Lie}(N) \to \text{Lie}(E) \to \text{Lie}(F) \to 0
\]

and this sequence is \( G \)-equivariant.

The bundle \( \text{Lie}(E) \) turns out to be the covariant Dieudonné module of \( F_0 \), so it is an “\( F \)-crystal”: it carries an integrable connection \( \nabla : \text{Lie}(E) \to \text{Lie}(E) \otimes \Omega^1_{A/\mathbb{C}} \) plus a “Frobenius structure”. The latter can be viewed as an isomorphism \( \sigma^* \text{Lie}(E) \to \text{Lie}(E) \) for any \( \sigma : A \to A \) lifting the \( q \)-power map on the special fibre. Let \( \mathcal{M} \) be the analytification of \( \text{Lie}(E) \), as a rigid vector bundle over \( X \); then by “Dwork’s trick”, \( \mathcal{M} \) admits a basis of horizontal sections over \( X \). (The idea: by formal integration, you get a basis over a small polydisc. But then you use the Frobenius pullback to “grow” this polydisc.) If you prefer, this can all be described in formulas in this case: see [GH2, Section 22].

We now have our representation \( V_K = H^0(X, \mathcal{M})^{\nabla} \) on the horizontal sections of \( \mathcal{M} \): the surjection \( \mathcal{M} \to \mathcal{L} \to 0 \) gives the map \( V_K \to H^0(X, \mathcal{L}) \) whose image is basepoint-free. The verification that \( \Phi \) is étale and detects isogenies can be found in [GH2, Section 23].

What is Dwork’s trick?

This is worth explaining a bit more, because it also comes up all over the place in \( p \)-adic cohomology. Say you have a vector bundle \( \mathcal{M} \) over the open unit polydisc \( X \) over \( K \) with coordinates \( x_1, \ldots, x_n \). (Note: what was \( n - 1 \) before is \( n \) now for notational simplicity.) It turns out that \( \mathcal{M} \) is in fact generated freely by global sections. [Correction of 27 Oct 2012:}

2
this was previously attributed to Gruson, which is incorrect. Instead see: W. Bartenwerfer, k-
holomorphe Vektorraumbündel auf offenen Polyzyllindern, J. reine angew Math. 326 (1981),
214-240.

Say you also have an integrable connection \( \nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1 \), i.e., commuting actions
of \( \partial_i = \frac{\partial}{\partial x_i} \) for \( i = 1, \ldots, n \). You can then try to formally solve for the horizontal sections
around \( x_1 = \cdots = x_n = 0 \). The resulting sections will converge on some polydisc, but its
radius may be much smaller than 1. E.g., if \( n = 1 \), the rank is 1, and \( \partial_t v = cv \), the horizontal
section is \( \exp(-\int c)v \), which converges on some disc but possibly a small one.

What having a Frobenius structure does is give you an isomorphism between \( \mathcal{M} \) and its
pullback along some map \( \sigma \) lifting the \( q \)-power Frobenius on the reduction of \( \Gamma(\mathcal{O}, X) \), e.g.,
one taking \( K \) into itself and taking \( x_i \) to \( x_i^q \). That pulls back your sections on a tiny polydisc
to sections on a larger polydisc (in my example, sections on the disc \( |x_i| \leq \rho \) pull back to
\( |x_i| \leq \rho^{1/q} \)); but in fact the \( K \)-vector space of horizontal sections is unique, so these sections
actually extend the ones you started with. Repeat ad infinitum.