Orthonormal bases

An orthogonal basis of a Banach space $V$ is a subset $\{e_i\}_{i \in I}$ of $V$ with the property that each $m \in M$ has a unique representation as a convergent sum $\sum_{i \in I} c_i e_i$, and if one always has $\|m\| = \max_{i \in I} \{\|c_i e_i\|\}$. (Note that “convergent” only makes sense if $I$ is at most countable; but note also that there is no distinction between “convergent” and “absolutely convergent” in the nonarchimedean setting!) The basis is orthonormal if $\|e_i\| = 1$ for each $i$. Lemma 1 from last time says that the norm on a Banach space of countable type can be approximated by equivalent norms which admit orthogonal bases; see below for an example.

Banach algebras

A Banach algebra (over $K$) is a Banach space $A$ over $K$ which is also a commutative $K$-algebra, and which satisfies the following additional restrictions.

(a) $\|1\| = 1$.

(b) for $x, y \in A$, $\|xy\| \leq \|x\| \cdot \|y\|$.

A Banach module over $A$ is a Banach space $M$ equipped with an $A$-module structure, such that $\|am\| \leq \|a\| \cdot \|m\|$ for $a \in A$ and $m \in M$.

Here’s an example where life turns out to be easier than in the real/complex case [FvdP, Lemma 1.2.3]. I’ll start next time with an example of this.

Lemma 1. Let $A$ be a Banach algebra over $K$ which is noetherian as a ring. Let $M$ be a Banach module over $A$ which is module-finite over $A$. Then any $A$-submodule of $M$ is closed.

Proof. Let $N$ be a submodule of $M$ and let $\tilde{N}$ be the closure of $N$. Since $A$ is noetherian, $\tilde{N}$ is module-finite over $A$; choose generators $e_1, \ldots, e_n$ of $\tilde{N}$ over $A$. Consider the $A$-module homomorphism $A^n \to \tilde{N}$ defined by $(a_1, \ldots, a_n) \mapsto \sum a_i e_i$, where $A^n$ is equipped with the norm $\|(a_1, \ldots, a_n)\| = \max_i \{|a_i|\}$. By the open mapping theorem, there exists $c \in (0,1)$ such that each $x \in \tilde{N}$ can be written as $\sum a_i e_i$ with $c \max_i \{|a_i|\} \leq \|x\|$. Choose $f_1, \ldots, f_n \in N$ with $\|e_i - f_i\| \leq c^2$; we show that $f_1, \ldots, f_n$ also generate $\tilde{N}$, which implies that $N = \tilde{N}$.

Given $x \in \tilde{N}$, define the sequence $x_0, x_1, \ldots$ as follows. Set $x_0 = x$; given $x_j$, write $x_j = \sum a_{j,i} e_i$ with $c \max_i \{|a_{j,i}|\} \leq \|x_j\|$, and put

$$x_{j+1} = \sum a_{j,i} (e_i - f_i),$$

so that $\|x_{j+1}\| \leq c \|x_j\|$. This means $x_j \to 0$ and so for each $i$, the series $\sum_j a_{j,i}$ converges to a limit $a_i$ satisfying $x = \sum a_i f_i$. Thus $N = \tilde{N}$, as desired. \qed
Tensor products

The tensor product of two Banach spaces $V$ and $W$ is not complete, so instead we will typically work with the completed tensor product $V \hat{\otimes} W$, which as the name suggests is just the completion of the ordinary tensor product as $K$-vector spaces.

The completed tensor product is also a Banach space, but this takes a bit of work to check. One gets a seminorm on $V \otimes W$ from the formula

$$\|x\| = \inf \{ \max_i \{ \|v_i\| \cdot \|w_i\| \} \},$$

the infimum taken over all presentations $x = \sum_{i=1}^m v_i \otimes w_i$, and this extends to the completion. To check that it’s a norm, one needs to check that if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of $V \otimes W$, with $x_j = \sum_{i=1}^m v_{ij} \otimes w_{ij}$, and $\max_i \{ \|v_{ij}\| \cdot \|w_{ij}\| \} \to 0$ as $j \to \infty$, then $x_j \to 0$ in $V \otimes W$. We may check this after replacing $V$ and $W$ by the span of the $v_{ij}$ and $w_{ij}$, respectively, dropping us into the countable type case. Then writing everything in terms of an orthogonal basis of an equivalent norm (using Lemma 1 from last time) yields the claim.

The completed tensor product is exact in the following sense. We say a sequence

$$0 \to M_1 \overset{f}{\to} M_2 \overset{g}{\to} M_3 \to 0$$

of maps between Banach spaces is exact if it is exact in the usual sense, and also $f$ and $g$ are isometric. (For $f$, this means the norm on $M_1$ is the restriction from $M_2$; for $g$, the norm on $M_3$ is the quotient norm from $M_2$.) Then for any Banach space $N$, the sequence

$$0 \to M_1 \hat{\otimes} N \overset{f}{\to} M_2 \hat{\otimes} N \overset{g}{\to} M_3 \hat{\otimes} N \to 0$$

is exact.

Warning: if $A$ is a Banach algebra, the product seminorm on a tensor product $M \otimes_A N$ of Banach modules $M$ and $N$ over $A$ may not be a norm! But see exercises for an important case where this is okay.

Exercises

1. (Leftover from last time) Prove that $\cup_{n=1}^\infty k((t^{1/n}))$ is algebraically closed for $k = \mathbb{C}$, but not for any field $k$ of positive characteristic. (Hint: look at $P(x) = x^p - x - t^{-1}$.)

2. Suppose $K$ is a complete discretely valued (ultrametric) field. Let $M$ be a Banach space over $K$ such that $\|m\| \in |K|$ for each $m \in M$. Put $\mathfrak{o}_M = \{ m \in M : \|m\| \leq 1 \}$ and $\overline{M} = \mathfrak{o}_M \otimes_K k$. Prove that a subset of $M$ forms an orthonormal basis if and only if its image in $\overline{M}$ is a basis of $\overline{M}$ as a $k$-vector space. (Hint: see [FvdP, Lemma 1.2.2].)

3. Let $A$ be a Banach algebra which is noetherian as a ring, and let $M$ and $N$ be Banach modules over $A$ which are module-finite over $A$. Show that the product seminorm

$$\|x\| = \inf \{ \max_i \{ \|m_i\| \cdot \|n_i\| : x = \sum_i m_i \otimes n_i \} \}$$

is a norm on $M \otimes N$, and that $M \otimes N$ is complete for this norm. (Hint: choose finite presentations of $M$ and $N$ and reduce to the case of free modules.)