Addenda on the spectral seminorm

A norm on a Banach algebra $A$ is power-multiplicative if $\|f^n\| = \|f\|^n$ for any $f \in A$ and any positive integer $n$. Our proof that the Gauss norm on $T_n$ has a topological characterization adapts to show that for any affinoid algebra $A$, there is at most one power-multiplicative Banach norm on $A$. We now know that if a power-multiplicative norm exists, it must be the spectral seminorm; hence such a norm exists if and only if $A$ is reduced.

In fact, the spectral seminorm is “minimal” in the following sense [BGR, Corollary 3.8.2/2].

**Proposition 1.** Let $A$ be an affinoid algebra with norm $\|\cdot\|$. Then for all $f \in A$, $\|f\|_{\text{spec}} \leq \|f\|$. In particular, $|f(x)| \leq \|f\|$ for any $x \in \text{Max } A$.

**Proof.** Apply the formula

$$\|f\|_{\text{spec}} = \lim_{n \to \infty} \|f^n\|^{1/n}$$

and note that $\|f^n\| \leq \|f\|^n$ because $\|\cdot\|$ is a Banach algebra norm.

This yields the following characterization of nilpotent elements, in the vein of our characterization of power-bounded elements [BGR, Proposition 6.2.3/2].

**Proposition 2.** For $A$ an affinoid algebra and $f \in A$, the following statements are equivalent:

(a) $f$ is topologically nilpotent (i.e., $\{f^n\}$ is a null sequence in $A$);

(b) $|f(x)| < 1$ for all $x \in \text{Max } A$;

(c) $\|f\|_{\text{spec}} < 1$.

**Proof.** The equivalence of (b) and (c) follows from the maximum modulus principle, and (a) implies (c) by the previous proposition. Given (c), choose $c \in K$ and $m \in \mathbb{N}$ such that $|c| > 1$ but $\|cf^m\|_{\text{spec}} \leq 1$. Then $cf^m$ is power-bounded (from last time), so $c^{-1}(cf^m) = f^m$ is topologically nilpotent, as then is $f$. Thus (c) implies (a), and we are done.

**Spectral norms are Banach norms**

We now know that the spectral seminorm on a reduced affinoid algebra is a norm. However, more than that is true: it is a Banach norm. (This proof is from [FvdP, Theorem 3.4.9]; the proof in [BGR, Theorem 6.2.4/1] is a bit more intricate.)

**Lemma 3.** Let $A \hookrightarrow B$ be an inclusion of affinoid algebras. Then the spectral seminorm on $B$ restricts to the spectral seminorm on $A$.
Proof. Choose a Banach norm $\| \cdot \|$ on $B$; it then restricts to a Banach norm on $A$, and applying the formula $\|f\|_{\text{spec}} = \lim_{n \to \infty} \|f^n\|^{1/n}$ gives us the claim. \qed

**Theorem 4.** Let $A$ be a reduced affinoid algebra. Then $A$ is complete under $\| \cdot \|_{\text{spec}}$. In particular, every Banach algebra norm on $A$ is equivalent to the spectral norm.

**Proof.** The last sentence will follow from what we showed earlier: any two Banach algebra norms on an affinoid algebra are equivalent. So we focus on showing that $A$ is complete.

We first reduce to the case where $A$ is an integral domain. Let $p_1, \ldots, p_m$ be the minimal primes of $A$. Choose a Banach norm $\| \cdot \|_A$ on $A$, put $A_i = A/p_i$ (which is an integral domain), and equip each $A_i$ with the quotient norm induced by $\| \cdot \|_A$. Then $A_1 \oplus \cdots \oplus A_m$ becomes a finitely generated Banach module over $A$ under the max norm

$$(a_1, \ldots, a_m) = \max_i \{\|a_i\|\}.$$  

Let $i : A \to A_1 \oplus \cdots \oplus A_m$ be the canonical injection; then $i(A)$ is an $A$-submodule of $A_1 \oplus \cdots \oplus A_m$. By the lemma from “$p$-adic functional analysis 2”, $i(A)$ is closed in $A_1 \oplus \cdots \oplus A_m$, and $i$ is an isomorphism onto its image by the open mapping theorem. The map $i$ is isometric for the spectral norms, so proving that the spectral norm on each $A_i$ is equivalent to the quotient norm proves that the spectral norm on $A$ is equivalent to $\| \cdot \|_A$.

From now on, we assume $A$ is an integral domain. If $B$ is a reduced affinoid algebra containing $A$, showing that $B$ is complete under its spectral seminorm implies that $A$ is complete under its spectral seminorm, thanks to the previous lemma. In particular, we can write $A$ as a finite integral extension of $T_d$ for some $d \geq 0$, and take $B$ to be the integral closure of $T_d$ in the normal closure of Frac $A$ over Frac $T_d$. This lets us break the problem into two steps.

(a) Show that if $A$ is finite over $T_d$ and Frac $A$ is purely inseparable over Frac $T_d$, then the spectral norm on $A$ is complete.

(b) Show that if $B$ is finite over $A$, Frac $B$ is Galois over Frac $A$, and the spectral norm on $A$ is fully multiplicative and complete, then the spectral norm on $B$ is complete.

So we work on these two steps separately.

(a) There is nothing to show unless the characteristic of $K$ is $p > 0$. Let $K'$ be the completed algebraic closure of $K$; then for some integer $m$,

$$A \subseteq K' \langle x_1^{1/p^m}, \ldots, x_d^{1/p^m} \rangle$$

so it’s enough to check the completeness of $K' \langle x_1^{1/p^m}, \ldots, x_d^{1/p^m} \rangle$ under its spectral norm. But again, this is true because that spectral norm is the Gauss norm.

(b) Put $G = \text{Gal}(\text{Frac } B / \text{Frac } A)$, and put

$$\text{Trace}(f) = \sum_{g \in G} f^g;$$
this gives a map from Frac $B$ to Frac $A$ such that $\| \text{Trace}(f) \|_{A,\text{spec}} \leq \| f \|_{B,\text{spec}}$. (This trace is the same as the trace of multiplication by $f$ as a Frac $A$-linear transformation on Frac $B$.) From basic algebra, we know that the Frac $A$-linear pairing

$$(x, y) \mapsto \text{Trace}(xy)$$

on Frac $B$ is nondegenerate.

Choose $e_1, \ldots, e_n \in B$ which form a basis for Frac $B$ over Frac $A$, and let $e_1^*, \ldots, e_n^*$ be the dual basis for the trace pairing. We show that $Ae_1 + \cdots + Ae_n$ is a Banach module under the spectral norm, by showing that the spectral norm is equivalent to the maximum norm

$$\| f_1 e_1 + \cdots + f_n e_n \|_{B,\text{spec}} = \max_i \{ \| f_i \|_{A,\text{spec}} \}. $$

Choose $a_0 \in A$ such that $a_0 e_j^* \in B^\text{spec}$ for $j = 1, \ldots, n$; now given $f_1 e_1 + \cdots + f_n e_n \in Ae_1 + \cdots + Ae_n$, we have

$$a_0 f_j = \text{Trace}(a_0 e_j^* \sum_i f_ie_i)$$

$$\| \text{Trace}(a_0 e_j^* \sum_i f_ie_i) \|_{A,\text{spec}} \leq \| a_0 e_j^* \sum_i f_ie_i \|_{B,\text{spec}}$$

$$\leq \| \sum_i f_ie_i \|_{B,\text{spec}}.$$

Therefore

$$\| \sum_i f_ie_i \|_{B,\text{spec}} \geq \| a_0 \|_{A,\text{spec}} \max_i \{ \| f_i \|_{A,\text{spec}} \}. $$

Since we also have

$$\| \sum_i f_ie_i \|_{B,\text{spec}} \leq \max_i \{ \| f_i \|_{A,\text{spec}} \} \max_i \{ \| e_i \|_{B,\text{spec}} \},$$

the spectral norm restricted to $Ae_1 + \cdots + Ae_n$ is equivalent to the maximum norm, and so is a Banach norm.

For some $a \in A$, $aB \subseteq Ae_1 + \cdots + Ae_n$; since the spectral norm on $A$ is multiplicative, the spectral norm on $B$ is thus complete.

Note that this theorem can also be interpreted as follows: a sequence of elements of $A$ converges to zero under some (any) Banach algebra norm if and only if it converges uniformly to zero on Max $A$. 

\[\square\]
The reduction of an affinoid algebra

In case you are wondering when the spectral seminorm is not just a norm but is actually fully multiplicative (like the Gauss norm), here is your answer. Recall that for \(A\) an affinoid algebra, we defined

\[
\mathfrak{a}_A^\text{spec} = \{ f \in A : \| f \|_{\text{spec}} \leq 1 \}.
\]

Now define

\[
\mathfrak{m}_A^\text{spec} = \{ f \in A : \| f \|_{\text{spec}} < 1 \}
\]

and \(\overline{A}^\text{spec} = \mathfrak{a}_A^\text{spec} / \mathfrak{m}_A^\text{spec} \); we call the latter the reduction of \(A\). Then we have the following [BGR, Proposition 6.2.3/5].

**Proposition 5.** The spectral seminorm is a fully multiplicative norm if and only if \(A\) is reduced and \(\overline{A}\) is an integral domain.

Note that \(A\) being an integral domain is not enough; see exercises.

**Proof.** We already know that the spectral seminorm is a norm if and only if \(A\) is reduced; also, if \(A\) is reduced and the spectral norm is fully multiplicative, then the product of elements of spectral norm 1 again has spectral norm 1, so \(\overline{A}\) is an integral domain. Conversely, suppose \(A\) is reduced and \(\overline{A}\) is an integral domain. Given \(f, g \in A\) nonzero, there exists an integer \(n\) such that \(\| f \|_{\text{spec}}^n\) and \(\| g \|_{\text{spec}}^n\) belong to \(|K^*|\), by the maximum modulus principle. (Namely, the spectral seminorm is always the norm of the evaluation of \(f\) at some point whose residue field is finite over \(K\).) Choose \(c, d \in K^*\) with \(c\| f \|_{\text{spec}}^n = d\| g \|_{\text{spec}}^n = 1\). Then the product of the images of \(cf^n\) and \(dg^n\) in \(\overline{A}\) must be nonzero because \(\overline{A}\) is an integral domain; that is, \(\| cf^n dg^n \|_{\text{spec}} = 1\). Hence

\[
1 = \| cf^n dg^n \|_{\text{spec}} = c\| f \|_{\text{spec}}^n \cdot d\| g \|_{\text{spec}}^n
\]

and (by power-multiplicativity of the spectral seminorm) it follows that \(\| fg \|_{\text{spec}} = \| f \|_{\text{spec}} \cdot \| g \|_{\text{spec}}\).

**Exercises**

1. Give an explicit example of an affinoid algebra \(A\) which is an integral domain, but whose spectral seminorm is not fully multiplicative. (Hint: consider power series in \(x\) and \(x^{-1}\) which converge on a suitable annulus; your motivation should functions on a complex annulus which have their suprema on opposite boundary components. We’ll look more at this geometric situation in the next few lectures.)