In this unit, we introduce (without proof for now) a formula which relates the distribution of primes to the zeroes of the Riemann zeta function. Given a suitable zero-free region for \( \zeta(s) \) in the critical strip, this can be used to prove the prime number theorem with an estimate for the error term.

1 Zeta zeroes and prime numbers

For \( x \notin \mathbb{N} \), define the counting function

\[
\psi(x) = \sum_{n \leq x} \Lambda(n),
\]

where \( \Lambda : \mathbb{N} \to \mathbb{R} \) is the von Mangoldt function

\[
\Lambda(n) = \begin{cases} 
\log p & n = p^a, a \geq 1 \\
0 & \text{otherwise.}
\end{cases}
\]

If \( x \in \mathbb{N} \), it is convenient to modify the definition to

\[
\psi(x) = \sum_{n < x} \Lambda(n) + \frac{1}{2} \Lambda(x).
\]

Note that for the function \( \vartheta \) we defined earlier as

\[
\vartheta(x) = \sum_{p \leq x} \log p,
\]

we have

\[
\psi(x) - \vartheta(x) = O(x^{1/2} \log x) \quad (x \to \infty)
\]

so the prime number theorem is equivalent to

\[
\psi(x) \sim x \quad (x \to \infty).
\]

The formula of von Mangoldt expresses the difference \( \psi(x) - x \) in terms of the zeroes of \( \zeta(s) \). We will prove this formula in a later unit.

**Theorem 1** (von Mangoldt’s formula). For \( x \geq 2 \) and \( T > 0 \),

\[
\psi(x) - x = - \sum_{\rho \mid \text{Im}(\rho) < T} \frac{x^\rho}{\rho} \zeta'(0) \zeta(0) - \frac{1}{2} \log(1 - x^{-2}) + R(x, T)
\]

with \( \rho \) running over the zeroes of \( \zeta(s) \) in the region \( \text{Re}(s) \in [0, 1] \), and

\[
R(x, T) = O\left( \frac{x \log^2(xT)}{T} + (\log x) \min \left\{ 1, \frac{x}{T(x)} \right\} \right).
\]

Here \( \langle x \rangle \) denotes the distance from \( x \) to the nearest prime power other than possibly \( x \) itself.
The region \( \text{Re}(s) \in [0, 1] \) is called the *critical strip* for \( \zeta \), because we can account for all of the zeroes outside this strip: they are the trivial zeroes \( s = -2, -4, \ldots \) forced by the functional equation and the fact that \( \Gamma(s/2) \) has poles at nonpositive even integers. In fact, the last term in the formula is merely \( -\sum_{\rho} \frac{x^\rho}{\rho} \) for \( \rho \) running over the trivial zeroes.

Incidentally, one can check by a numerical calculation that there are no real zeroes of \( \zeta \) in the critical strip, by numerically approximating the integral representation of \( \xi(s) \). This raises an interesting point: in general, direct numerical approximation can be used to prove that an analytic function does not vanish in a region, but not that it does vanish at a particular point. The best one can do is use a zero-counting formula to prove that there must be a zero near the proposed vanishing point.

Note that for \( x \) fixed, \( R(x, T) = o(1) \) as \( T \to \infty \), so we have

\[
\psi(x) - x = -\sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2})
\]

as long as we interpret the sum over \( \rho \) to mean the limit of the partial sums over \( |\text{Im}(\rho)| < T \) as \( T \to \infty \). This formula, while pretty, is not as useful in practice as the form with remainder; we will use the remainder form by taking \( T \) to be some (preferably large) function of \( x \) as \( x \to \infty \).

2 How to use von Mangoldt’s formula

In order to use von Mangoldt’s formula to bound \( \psi(x) - x \), we need to give an upper bound on the sum \( \sum_{\rho} \frac{x^\rho}{\rho} \) for \( \rho \) running over nontrivial zeroes of \( \zeta \) in the region \( |\text{Im}(s)| \leq T \).

Put \( \beta = \text{Re}(\rho), \gamma = \text{Im}(\rho) \). Suppose we can prove that \( \beta < 1 - f(|\gamma|) \) for some nonincreasing function \( f : [0, \infty) \to (0, 1/2) \); then

\[
|x^\rho| = x^\beta < x^{1-f(|\gamma|)} < x^{1-f(T)}
\]

and \( |\rho| \geq |\gamma| \). We thus have

\[
\left| \sum_{\rho:|\gamma|<T} \frac{x^\rho}{\rho} \right| \leq x^{1-f(T)} \sum_{\rho:|\gamma|<T} \frac{1}{|\gamma|}.
\]

Let \( N(T) \) be the number of zeroes in the critical strip with \( |\gamma| \leq T \). Then

\[
\sum_{\rho:0<|\gamma|<T} \frac{1}{|\gamma|} = \int_0^T t^{-1}dN(t) = \frac{N(T)}{T} + \int_0^T t^{-2}N(t) dt.
\]

At this point we need some information about \( N(T) \); again, we will prove this (and a bit more) later.

**Theorem 2** (Hadamard). We have \( N(T) = O(T \log T) \) as \( T \to \infty \).
This implies that
\[ \left| \sum_{\rho: \gamma < T} \frac{1}{\rho} \right| = O(\log^2 T), \]
so
\[ \left| \sum_{\rho: \gamma < T} \frac{x^\rho}{\rho} \right| = O(x^{1-f(T)} \log^2 T). \]

For $x$ an integer, we now take $T = T(x)$ to be a suitable function of $x$, and invoke von Mangoldt’s formula with remainder to deduce that
\[ \psi(x) - x = O \left( x^{1-f(T)} \log^2 T(x) + \frac{x \log^2 x}{T(x)} + \frac{x \log^2 T(x)}{T(x)} \right). \quad (1) \]

3 The Riemann Hypothesis

Riemann calculated a few of the zeroes of $\zeta$ and, based on this evidence, made the following remarkable conjecture (whose resolution is worth $1,000,000 from the Clay Mathematics Institute).

**Conjecture 3** (Riemann Hypothesis). The nontrivial zeroes of $\zeta$ all lie on the line $\text{Re}(s) = \frac{1}{2}$.

This is a best-case scenario in terms of deducing error bounds on $\psi(x) - x$. Namely, suppose every nontrivial zero $\rho$ of $\zeta$ satisfies $c \leq \text{Re}(\rho) \leq 1 - c$ for some $c \in (0, 1/2)$; then we can take $f(T) = c$ in (1), yielding
\[ \psi(x) - x = O \left( x^{1-c} \log^2 T(x) + \frac{x \log^2 x}{T(x)} + \frac{x \log^2 T(x)}{T(x)} \right). \]

By taking $T(x) = x$, we obtain
\[ \psi(x) - x = O(x^{1-c} \log^2 x). \]
If I can take $c$ to be any value less than $1/2$, that means
\[ \psi(x) - x = O(x^{1/2 + \epsilon}) \quad (\epsilon > 0), \]
and similarly one gets a strong estimate on $\pi(x)$ (see exercises).

Unfortunately, for no value of $c > 0$ are we able at present to prove that every nontrivial zero $\rho$ satisfies $\text{Re}(\rho) \leq 1 - c$. We will give a much smaller zero-free region in a later unit.
4 Variants for $L$-functions

For $\chi$ a Dirichlet character, define

$$\psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n),$$

where again we multiply the $n = x$ term by $1/2$ if it is present.

**Theorem 4.** For $\chi$ a nonprincipal Dirichlet character of level $N$,

$$\psi(x, \chi) = -\sum_{\rho \in \gamma} \frac{x^\rho}{\rho} - (1 - a) \log x - b(\chi) + \sum_{m=1}^{\infty} \frac{x^{a-2m}}{2m-a} + R(x, T),$$

where $b(\chi)$ is an explicit constant, $a = 1$ for $\chi$ even and $a = 0$ for $\chi$ odd, and

$$R(x, T) = O\left(\frac{x \log^2(NxT)}{T} + (\log x) \min\left\{1, \frac{x}{T(x)}\right\}\right).$$

For a fixed $N$, one can use this formula together with a zero-free region for all of the $L(s, \chi)$ with $\chi$ of level $N$, to obtain a prime number theorem for arithmetic progressions of difference $N$ with an estimate for the error term.

However, one would also like to be able to establish a prime number theorem with error term for arithmetic progressions where the difference is allowed to vary. In this case, one of course must have a zero-free region for all of the relevant characters. But there are two extra complications.

- One must understand how the constant $b(\chi)$ varies with $\chi$.
- One must deal with possible roots of $L(s, \chi)$ that are very close to $s = 0$ or $s = 1$ (so-called Siegel zeroes).

Dealing with these goes beyond the level of detail I have in mind for this course; see Davenport §14–22 for a systematic exposition.

**Exercises**

1. Assume that $\psi(x) = x + o(x^{1-\epsilon})$ for some given $\epsilon \in (0, 1/2)$. Deduce a corresponding upper bound for $\pi(x) - \text{li}(x)$, where $\text{li}(x)$ is the logarithmic integral function

$$\text{li}(x) = \int_2^x \frac{dt}{\log t}.$$

Then deduce that

$$\pi(x) - \frac{x}{\log x} \neq o(x^{1-\delta})$$

for any $\delta > 0$. (This last statement can be proved unconditionally, but don’t worry about that for now.) This is the sense in which $\text{li}(x)$ is a better approximation than $x/(\log x)$ of the count of primes.