In this unit, we establish the functional equation property for the Riemann zeta function, which will imply its meromorphic continuation to the entire complex plane. We will do likewise for Dirichlet $L$-functions in the next unit.

1 The functional equation for $\zeta$

A “random” Dirichlet series $\sum_n a_n n^{-s}$ will not exhibit very interesting analytic behavior beyond its abscissa of absolute convergence. However, we already know that $\zeta$ is atypical in this regard, in that we can extend it at least as far as $\Re(s) > 0$ if we allow the simple pole at $s = 1$. One of Riemann’s key observations is that in the strip $0 < \Re(s) < 1$, $\zeta$ obeys a symmetry property relating $\zeta(s)$ to $\zeta(1 - s)$; once we prove this, we will then be able to extend $\zeta$ all the way across the complex plane. (This is essentially Riemann’s original proof; several others are possible.)

We first recall the definition and basic properties of the $\Gamma$ function. We may define

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

for $\Re(s) > 0$. It is then straightforward to check (by integration by parts) that

$$\Gamma(s + 1) = s\Gamma(s) \quad (\Re(s) > 0).$$

(1)

Using (1), we may extend $\Gamma$ to a meromorphic function on all of $\mathbb{C}$, with simple poles at $s = 0, -1, \ldots$. Since $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$, we have that for $n$ a nonnegative integer,

$$\Gamma(n + 1) = n!.$$

Substituting $t = \pi n^2 x$ in the definition of $\Gamma$, we have

$$\pi^{-s/2} \Gamma(s/2) n^{-s} = \int_0^\infty x^{s/2-1} e^{-n^2 \pi x} dx \quad \Re(s) > 0.$$

If we sum over $n$, we can interchange the sum and integral for $\Re(s) > 1$ because the sum-integral converges absolutely. Hence

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_0^\infty x^{s/2-1} \omega(x) dx \quad \Re(s) > 1$$

for

$$\omega(x) = \sum_{n=1}^\infty e^{-n^2 \pi x}.$$
It is slightly more convenient to work with the function $\theta$ defined by

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x},$$

which clearly satisfies $2\omega(x) = \theta(x) - 1$.

At this point, Riemann recognized $\theta$ as a function of the sort considered by Jacobi in the late 19th century; from that work, Riemann knew about the identity

$$\theta(x^{-1}) = x^{1/2} \theta(x) \quad x > 0. \quad (2)$$

We will return to the proof of (2) in the next section; for the moment, let’s see how we use this to get a functional equation for $\zeta$.

Returning to

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_0^{\infty} x^{s/2 - 1} \omega(x) \, dx,$$

we take the natural step of splitting the integral at $x = 1$, then substituting $1/x$ for $x$ in the integral from 0 to 1. This yields

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_1^{\infty} x^{s/2 - 1} \omega(x) \, dx + \int_1^{\infty} x^{-s/2 - 1} \omega(1/x) \, dx.$$

From (2), we deduce

$$\omega(x^{-1}) = -\frac{1}{2} + \frac{1}{2} x^{1/2} + x^{1/2} \omega(x),$$

yielding

$$\int_1^{\infty} x^{-s/2 - 1} \omega(x^{-1}) \, dx = -\frac{1}{s} + \frac{1}{s - 1} + \int_1^{\infty} x^{-s/2 - 1/2} \omega(x) \, dx$$

and so

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = -\frac{1}{s(1 - s)} + \int_1^{\infty} (x^{s/2 - 1} + x^{(1-s)/2 - 1}) \omega(x) \, dx, \quad (3)$$

at least for $\text{Re}(s) > 1$.

But now notice that the left side of (3) represents a meromorphic function on $\text{Re}(s) > 0$, whereas the right side of (3) represents a meromorphic function on all of $\mathbb{C}$, because the integral converges absolutely for all $z$. (That’s because $\omega(x) = O(e^{-\pi x})$ as $x \to +\infty$.)

This has tons of consequences. First, (3) is also valid for $\text{Re}(s) > 0$. Second, we can use (3) to define $\zeta(s)$ as a meromorphic function on all of $\mathbb{C}$. Third, the right side of (3) is invariant under the substitution $s \mapsto 1 - s$, so we obtain a functional equation for $\zeta$. One often writes this by defining

$$\xi(s) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma(s/2) \zeta(s),$$

and then the functional equation is $\xi(1 - s) = \xi(s)$. Fourth, the function $\xi(s)$ is actually entire for $\text{Re}(s) > 0$ because the factor of $s - 1$ counters the pole of $\zeta$ at $s = 1$; by the functional equation, $\xi$ is entire everywhere.
Remember that $\zeta(s)$ has no zeroes in the region $\text{Re}(s) \geq 1$. By the functional equation, in the region $\text{Re}(s) \leq 0$, the only zeroes of $\zeta$ occur at the poles of $\Gamma(s/2)$ (except for $s = 0$, where the factor of $s$ counts the pole), i.e., at negative even integers. These are called the trivial zeroes of $\zeta$; the other zeroes, which are forced to lie in the range $0 < \text{Re}(s) < 1$, are much more interesting!

## 2 The $\theta$ function and the Fourier transform

I still owe you the functional equation (2) for the $\theta$ function. It is usually deduced from the Poisson summation formula for Fourier transforms, which I'll now recall.

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be an integrable ($L^1$) function. The Fourier transform of $f$ is then defined as the function $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$ given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t) \, dt;$$

it is uniformly continuous.

It is convenient to restrict attention to a smaller class of functions. We say $f : \mathbb{R} \rightarrow \mathbb{C}$ is a Schwarz function if $f$ is infinitely differentiable and, for each nonnegative integer $n$ and each $c \in \mathbb{R}$, $|f^{(n)}(t)| = o(|t|^c)$ as $t \rightarrow \pm \infty$.

**Lemma 1.** Let $f, g : \mathbb{R} \rightarrow \mathbb{C}$ be Schwarz functions.

(a) The functions $\hat{f}, \hat{g}$ are again Schwarz functions.

(b) We have $\hat{f}(t) = f(-t)$.

(b) If we define the convolution $f \ast g$ by

$$(f \ast g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u) \, du,$$

then $\hat{f} \ast g(s) = \hat{f}(s)\hat{g}(s)$.

**Proof.** Exercise. $\square$

**Theorem 2** (Poisson summation formula). Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a Schwarz function. Then

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n); \quad (4)$$

in particular, the sum on the right converges.
Sketch of proof. One can give a simple proof using Fourier series; see exercises. Here I’ll sketch a more direct approach which has some significance in analytic number theory; it is a very simple version of the Hardy-Littlewood “circle method”.

Write

\[
\sum_{n=-N}^{N} \hat{f}(n) = \int_{-\infty}^{+\infty} \sum_{n=-N}^{n} e^{-2\pi int} f(t) \, dt \\
= \sum_{m=\infty}^{\infty} \int_{m-1/N}^{m+1/N} \frac{e^{-2\pi i Nt} - e^{2\pi i(N+1)t}}{1 - e^{2\pi it}} f(t) \, dt \\
+ \sum_{m=\infty}^{\infty} \int_{m+1/N}^{m+1-1/N} \frac{e^{-2\pi i Nt} - e^{2\pi i(N+1)t}}{1 - e^{2\pi it}} f(t) \, dt.
\]

Then check that the summand in the first sum converges (uniformly on \(m\)) to \(f(m)\), while the second summand converges (uniformly on \(m\)) to zero. (If you prefer, first use a partition of unity to reduce to the case where \(f\) is supported on an interval like \([-2/3, 2/3]\), so that the sums over \(m\) become finite.)

The Poisson summation formula immediately yields (2) as soon as one checks that \(f(t) = e^{-\pi t^2}\) is invariant under the Fourier transform (see exercises): it then follows that the Fourier transform of \(f(t) = e^{-\pi xt^2}\) is \(\hat{f}(s) = x^{-1/2} e^{-\pi x s^2}\), and Poisson summation gives (2).

2.1 Asides

Our study of \(\theta\) merely grazes the top of a very large iceberg. Here are three comments to this effect.

A much more general version of \(\theta\) was considered by Jacobi, in which he considered a quadratic form \(Q(x_1, \ldots, x_m)\) and formed the sum

\[
\theta_Q(x) = \sum_{n_1, \ldots, n_m \in \mathbb{Z}} e^{-Q(n_1, \ldots, n_m)\pi x},
\]

if \(Q\) is positive definite, this again converges rapidly for all \(x\).

One can also think of \(\theta\) as a example of a special sort of complex function called a modular form. Nowadays, modular forms are central not just to analytic number theory, but a lot of algebraic number theory as well. For instance, the “modularity of elliptic curves” is central to the proof of Fermat’s last theorem; I may say a bit about this later in the course.

The Fourier transform is a typical example of an integral transform; the function \(s, t \mapsto e^{-2\pi ist}\) is the kernel of this transform. Another important integral transform in analytic number theory is the Mellin transform: for a function \(f : [0, \infty) \to \mathbb{C}\), the Mellin transform \(M(f)\) is given by

\[
M(f)(s) = \int_{0}^{\infty} f(t)t^{s-1} \, dt.
\]

For instance, \(\Gamma(s)\) is the Mellin transform of \(e^{-t}\).
Exercises

1. Rewrite the functional equation directly in terms of $\zeta(s)$ and $\zeta(1 - s)$.

2. What is the residue of the pole of $\Gamma$ at a nonpositive integer $s$?

3. Prove Lemma 1.

4. Prove the Poisson summation formula either by completing the sketch given above, or by considering the Fourier series of the function

$$F(s) = \sum_{m \in \mathbb{Z}} f(s + m).$$

5. Prove that the function $f(t) = e^{-\pi t^2}$ is its own Fourier transform.