18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
The functional equation for the Riemann zeta function

In this unit, we establish the functional equation property for the Riemann zeta function, which will imply its meromorphic continuation to the entire complex plane. We will do likewise for Dirichlet L-functions in the next unit.

1 The functional equation for ζ

A "random" Dirichlet series $\sum_n a_n n^{-s}$ will not exhibit very interesting analytic behavior beyond its abscissa of absolute convergence. However, we already know that ζ is atypical in this regard, in that we can extend it at least as far as $\mathrm{Re}(s) > 0$ if we allow the simple pole at $s = 1$. One of Riemann's key observations is that in the strip $0 < \mathrm{Re}(s) < 1$, ζ obeys a symmetry property relating $\zeta(s)$ to $\zeta(1-s)$; once we prove this, we will then be able to extend ζ all the way across the complex plane. (This is essentially Riemann's original proof; several others are possible.)

We first recall the definition and basic properties of the Γ function. We may define

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt$$

for $\mathrm{Re}(s) > 0$. It is then straightforward to check (by integration by parts) that

$$\Gamma(s+1) = s\Gamma(s) \qquad (\mathrm{Re}(s) > 0). \tag{1}$$

Using (1), we may extend Γ to a meromorphic function on all of $\mathbb{C}$, with simple poles at $s = 0, -1, \dots$. Since $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$, we have that for n a nonnegative integer,

$$\Gamma(n+1) = n!.$$

Substituting $t = \pi n^2 x$ in the definition of Γ, we have

$$\pi^{-s/2}\Gamma(s/2) n^{-s} = \int_0^\infty x^{s/2-1} e^{-n^2\pi x}\,dx \qquad \mathrm{Re}(s) > 0.$$

If we sum over n, we can interchange the sum and integral for $\mathrm{Re}(s) > 1$ because the sum-integral converges absolutely. Hence

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty x^{s/2-1}\omega(x)\,dx \qquad \mathrm{Re}(s) > 1$$

for

$$\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}.$$

It is slightly more convenient to work with the function θ defined by

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x},$$

which clearly satisfies $2\omega(x) = \theta(x) - 1$.

At this point, Riemann recognized θ as a function of the sort considered by Jacobi in the late 19th century; from that work, Riemann knew about the identity

$$\theta(x^{-1}) = x^{1/2}\theta(x) \qquad x > 0. \tag{2}$$

We will return to the proof of (2) in the next section; for the moment, let's see how we use this to get a functional equation for ζ.

Returning to

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty x^{s/2-1}\omega(x)\,dx,$$

we take the natural step of splitting the integral at $x = 1$, then substituting $1/x$ for x in the integral from 0 to 1. This yields

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_1^\infty x^{s/2-1}\omega(x)\,dx + \int_1^\infty x^{-s/2-1}\omega(1/x)\,dx.$$

From (2), we deduce

$$\omega(x^{-1}) = -\frac{1}{2} + \frac{1}{2}x^{1/2} + x^{1/2}\omega(x),$$

yielding

$$\int_1^\infty x^{-s/2-1}\omega(x^{-1})\,dx = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty x^{-s/2-1/2}\omega(x)\,dx$$

and so

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = -\frac{1}{s(1-s)} + \int_1^\infty (x^{s/2-1} + x^{(1-s)/2-1})\omega(x)\,dx, \tag{3}$$

at least for $\mathrm{Re}(s) > 1$.

But now notice that the left side of (3) represents a meromorphic function on $\mathrm{Re}(s) > 0$, whereas the right side of (3) represents a meromorphic function on all of $\mathbb{C}$, because the integral converges absolutely for all z. (That's because $\omega(x) = O(e^{-\pi x})$ as $x \to +\infty$.)

This has tons of consequences. First, (3) is also valid for $\mathrm{Re}(s) > 0$. Second, we can use (3) to *define* $\zeta(s)$ as a meromorphic function on all of $\mathbb{C}$. Third, the right side of (3) is invariant under the substitution $s \mapsto 1 - s$, so we obtain a functional equation for ζ. One often writes this by defining

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

and then the functional equation is $\xi(1-s) = \xi(s)$. Fourth, the function $\xi(s)$ is actually entire for $\mathrm{Re}(s) > 0$ because the factor of $s - 1$ counters the pole of ζ at $s = 1$; by the functional equation, ξ is entire everywhere.

Remember that $\zeta(s)$ has no zeroes in the region $\mathrm{Re}(s) \geq 1$. By the functional equation, in the region $\mathrm{Re}(s) \leq 0$, the only zeroes of ζ occur at the poles of $\Gamma(s/2)$ (except for $s = 0$, where the factor of s counters the pole), i.e., at negative even integers. These are called the *trivial zeroes* of ζ; the other zeroes, which are forced to lie in the range $0 < \mathrm{Re}(s) < 1$, are much more interesting!

2 The θ function and the Fourier transform

I still owe you the functional equation (2) for the θ function. It is usually deduced from the Poisson summation formula for Fourier transforms, which I'll now recall.

Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable (L^1) function. The *Fourier transform* of f is then defined as the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt;$$

it is uniformly continuous.

It is convenient to restrict attention to a smaller class of functions. We say $f : \mathbb{R} \to \mathbb{C}$ is a *Schwarz function* if f is infinitely differentiable and, for each nonnegative integer n and each $c \in \mathbb{R}$, $|f^{(n)}(t)| = o(|t|^c)$ as $t \to \pm\infty$.

Lemma 1. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be Schwarz functions.*

(a) The functions $\hat{f}, \hat{g}$ are again Schwarz functions.

(a) We have $\hat{\hat{f}}(t) = f(-t)$.

(b) If we define the convolution $f \star g$ by

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(t-u) g(u)\, du,$$

then $\widehat{f \star g}(s) = \widehat{f}(s)\widehat{g}(s)$.

Proof. Exercise. □

Theorem 2 (Poisson summation formula)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwarz function. Then*

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n); \tag{4}$$

in particular, the sum on the right converges.

Sketch of proof. One can give a simple proof using Fourier series; see exercises. Here I'll sketch a more direct approach which has some significance in analytic number theory; it is a very simple version of the Hardy-Littlewood "circle method".

Write

$$\begin{aligned}\sum_{n=-N}^{N} \hat{f}(n) &= \int_{-\infty}^{+\infty} \sum_{n=-N}^{n} e^{-2\pi i n t} f(t)\,dt \\ &= \sum_{m=-\infty}^{\infty} \int_{m-1/N}^{m+1/N} \frac{e^{-2\pi i N t} - e^{2\pi i (N+1) t}}{1 - e^{2\pi i t}} f(t)\,dt \\ &\quad + \sum_{m=-\infty}^{\infty} \int_{m+1/N}^{m+1-1/N} \frac{e^{-2\pi i N t} - e^{2\pi i (N+1) t}}{1 - e^{2\pi i t}} f(t)\,dt.\end{aligned}$$

Then check that the summand in the first sum converges (uniformly on m) to $f(m)$, while the second summand converges (uniformly on m) to zero. (If you prefer, first use a partition of unity to reduce to the case where f is supported on an interval like $[-2/3, 2/3]$, so that the sums over m become finite.) $\square$

The Poisson summation formula immediately yields (2) as soon as one checks that $f(t) = e^{-\pi t^2}$ is invariant under the Fourier transform (see exercises): it then follows that the Fourier transform of $f(t) = e^{-\pi x t^2}$ is $\hat{f}(s) = x^{-1/2} e^{-\pi x s^2}$, and Poisson summation gives (2).

2.1 Asides

Our study of θ merely grazes the top of a very large iceberg. Here are three comments to this effect.

A much more general version of θ was considered by Jacobi, in which he considered a quadratic form $Q(x_1, \dots, x_m)$ and formed the sum

$$\theta_Q(x) = \sum_{n_1,\dots,n_m \in \mathbb{Z}} e^{-Q(n_1,\dots,n_m)\pi x};$$

if Q is positive definite, this again converges rapidly for all x.

One can also think of θ as a example of a special sort of complex function called a *modular form*. Nowadays, modular forms are central not just to analytic number theory, but a lot of algebraic number theory as well. For instance, the "modularity of elliptic curves" is central to the proof of Fermat's last theorem; I may say a bit about this later in the course.

The Fourier transform is a typical example of an *integral transform*; the function $s, t \mapsto e^{-2\pi i s t}$ is the *kernel* of this transform. Another important integral transform in analytic number theory is the *Mellin transform*: for a function $f : [0, \infty) \to \mathbb{C}$, the Mellin transform $M(f)$ is given by

$$M(f)(s) = \int_0^{\infty} f(t) t^{s-1}\,dt.$$

For instance, $\Gamma(s)$ is the Mellin transform of e^{-t}.

Exercises

1. Rewrite the functional equation directly in terms of $\zeta(s)$ and $\zeta(1-s)$.
2. What is the residue of the pole of Γ at a nonpositive integer s?
3. Prove Lemma 1.
4. Prove the Poisson summation formula either by completing the sketch given above, or by considering the Fourier series of the function
$$F(s) = \sum_{m \in \mathbb{Z}} f(s+m).$$
5. Prove that the function $f(t) = e^{-\pi t^2}$ is its own Fourier transform.