18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
The functional equations for Dirichlet L-functions

In this unit, we establish the functional equation property for Dirichlet L-functions. Much of the work is left as exercises.

1 Even characters

Let χ be a Dirichlet character of level N. We say χ is *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$.

For χ even, we can derive a functional equation for $L(s, \chi)$ by imitating the argument we used for ζ. Start with

$$\chi(n)\pi^{-s/2}N^{s/2}\Gamma(s/2)n^{-s} = \int_0^\infty \chi(n)e^{-\pi n^2 x/N}x^{s/2-1}\,dx$$

and sum over n to obtain

$$\pi^{-s/2}N^{s/2}\Gamma(s/2)L(s,\chi) = \frac{1}{2}\int_0^\infty x^{s/2-1}\theta(x,\chi)\,dx \tag{1}$$

for

$$\theta(x,\chi) = \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 x/N}.$$

(Notice there is no additive constant because $\chi(0) = 0$.)

Applying the Poisson summation formula to $\theta(x, \chi)$ looks problematic, because $\chi(n)$ does not extend nicely to a function on all of $\mathbb{R}$. Fortunately we can avoid this by doing a bit more Fourier analysis, but this time on the *additive* group $\mathbb{Z}/N\mathbb{Z}$: write

$$\chi(n) = \sum_{m=1}^{N} c_{\chi,m}e^{2\pi imn/N}$$

with

$$c_{\chi,m} = \frac{1}{N}\sum_{l=1}^{N}\chi(l)e^{-2\pi ilm/N}.$$

I'll come back to what this quantity $c_{\chi,m}$ actually is in a moment. In the meantime, let's see what happens when we use this new expression for $\chi(n)$. Or rather, I'll let you see what happens as an exercise; you should get

$$\theta(x,\chi) = (N/x)^{1/2}\sum_{m=1}^{N} c_{\chi,m}\sum_{n=-\infty}^{\infty} e^{-\pi(nN+m)^2/(xN)}. \tag{2}$$

To get further, we need some description of the $c_{\chi,m}$ which is somehow uniform in m. Here it is: if m is coprime to N, then

$$\begin{aligned} c_{\chi,m} &= \frac{1}{N}\sum_{l=1}^{N} \chi(l) e^{-2\pi i l m/N} \\ &= \frac{1}{N}\sum_{l=1}^{N} \overline{\chi(m)}\chi(lm) e^{-2\pi i l m/N} \\ &= \overline{\chi}(m) c_{\chi,1}. \end{aligned}$$

For m not coprime to N, we must assume χ is primitive, and then again

$$c_{\chi,m} = \overline{\chi}(m) c_{\chi,1} \tag{3}$$

but this is not so obvious; see exercises.

This gives us

$$\theta(x,\chi) = (N/x)^{1/2} c_{\chi,1}\theta(x^{-1},\overline{\chi}),$$

and now we are home free: again split the integral (1) at 1 and substitute $x \mapsto x^{-1}$ in one term, to obtain

$$\begin{aligned} \pi^{-s/2}N^{s/2}\Gamma(s/2)L(s,\chi) &= \frac{1}{2}\int_1^\infty x^{s/2-1}\theta(x,\chi)\,dx + \frac{1}{2}\int_1^\infty x^{-s/2-1}\theta(x^{-1},\chi)\,dx \\ &= \frac{1}{2}\int_1^\infty x^{s/2-1}\theta(x,\chi)\,dx + \frac{1}{2}N^{1/2}c_{\chi,1}\int_1^\infty x^{(1-s)/2-1}\theta(x,\overline{\chi})\,dx. \end{aligned}$$

Similarly,

$$\pi^{-s/2}N^{s/2}\Gamma(s/2)L(s,\overline{\chi}) = \frac{1}{2}\int_1^\infty x^{s/2-1}\theta(x,\overline{\chi})\,dx + \frac{1}{2}N^{1/2}c_{\overline{\chi},1}\int_1^\infty x^{(1-s)/2-1}\theta(x,\chi)\,dx.$$

It is elementary to check that $c_{\chi,1}c_{\overline{\chi},1} = N^{-1}$ (see exercises); we thus obtain

$$\pi^{-(1-s)/2}N^{(1-s)/2}\Gamma((1-s)/2)L(1-s,\overline{\chi}) = N^{1/2}c_{\overline{\chi},1}\pi^{-s}N^{s/2}\Gamma(s/2)L(s,\chi). \tag{4}$$

Again, the extra factors of π, N, Γ should be thought of as an "extra Euler factor" coming from the "prime at infinity".

Pay close attention to the fact that unless $\chi = \overline{\chi}$, the functional equation 4 relates two *different* L-functions. In a few circumstances, this makes it less useful than if it related a single $L(s,\chi)$ to itself, but so be it.

Also note that quantity $c_{\chi,1}$ is related to the more commonly introduced *Gauss sum* associated to χ:

$$\tau(\chi) = Nc_{\overline{\chi},1} = \sum_{l=1}^{N} \chi(l) e^{2\pi i l/N}.$$

For more about Gauss sums, see the exercises.

2 Odd characters

We have to do something different if $\chi(-1) = -1$, as then the function $\theta(x, \chi)$ as defined above is identically zero. Instead we use

$$\theta_1(x, \chi) = \sum_{n=\infty}^{\infty} n\chi(n)e^{-n^2\pi x/N}$$

and shift s around a bit. Namely,

$$\pi^{-(s+1)/2}N^{(s+1)/2}\Gamma((s+1)/2)L(s, \chi) = \frac{1}{2}\int_0^\infty \theta_1(x, \chi)x^{(s+1)/2-1}\,dx.$$

Again you split the integral at $x = 1$ and use an inversion formula; this time the right identity is

$$\sum_{n=-\infty}^{\infty} ne^{-n^2\pi x/N+2\pi imn/N} = i(N/x)^{3/2}\sum_{n=-\infty}^{\infty}\left(n + \frac{m}{N}\right)e^{-\pi(n+m/N)^2N/x}. \tag{5}$$

You should end up with the functional equation

$$\pi^{-(2-s)/2}N^{(2-s)/2}\Gamma((2-s)/2)L(1-s, \overline{\chi}) = -i\tau(\overline{\chi})N^{-1/2}\pi^{-(1+s)/2}N^{(1+s)/2}\Gamma((1+s)/2)L(s, \chi). \tag{6}$$

Since this is now the third time through this manner of argument, I leave further details to the exercises.

Exercises

1. Prove the following functional equations for Γ:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$
$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\pi^{1/2}\Gamma(2s).$$

   Then use these to give a simplified functional equation for ζ of the form "$\zeta(1-s)$ equals $\zeta(s)$ times some explicit function".

2. Prove that (3) holds for χ primitive whether or not m is coprime to N.

3. Prove that for χ primitive, $\tau(\chi)\overline{\tau(\chi)} = N$. (Warning: the value of $\tau(\chi)\tau(\overline{\chi})$ depends on whether χ is even or odd.) Then exhibit an example where this fails if χ is imprimitive.

4. For χ a Dirichlet character of level N, based on the functional equation, where does $L(s, \chi)$ have zeroes and poles in the region $\mathrm{Re}(s) \leq 0$?

5. Prove that

$$\sum_{n=-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} = x^{1/2} \sum_{n=-\infty}^{\infty} e^{-n^2\pi x + 2\pi i n\alpha} \qquad (\alpha \in \mathbb{R}, x > 0);$$

then prove (5) by the same method (namely Poisson summation).

6. Use the previous exercise to deduce (2).

7. Prove the functional equation (6).

8. Pick an example of a nonprincipal nonprimitive character χ, and write out the functional equation for $L(s, \chi)$.

9. (Dirichlet) For $a, b \in \mathbb{Z}$ and $f : \mathbb{R} \to \mathbb{C}$ a function obtained by taking a continuous function on $[a, b]$ and setting its other values to 0, the Poisson summation formula still holds if interpreted as

$$\frac{1}{2}f(a) + f(a+1) + \cdots + f(b-1) + \frac{1}{2}f(b) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

(you don't have to prove this). Apply this to the function

$$f(t) = \begin{cases} e^{2\pi i t^2/N} & t \in [0, N] \\ 0 & \text{otherwise} \end{cases}$$

in order to evaluate $\sum_{n=1}^{N} e^{2\pi i n^2/N}$ for N a positive integer. Then use this to compute $G(\chi)$ for χ the quadratic character $\chi(m) = \left(\frac{m}{p}\right)$. (Optional, not to be turned in: give a more elementary computation of $G(\chi)^2$.)