18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
Small gaps between primes (proofs)

Here are the missing calculations from the Goldston-Pintz-Yıldırım theorems. The reference is the article by Goldston et al cited in the previous unit.

1 Review of notation

Fix once and for all positive integers k, ℓ. Let x be a parameter tending to ∞. Let $\mathcal{H} = (h_1, \dots, h_k)$ be a k-tuple of distinct integers in the range $1, \dots, H$, where $H \leq \lambda \log x$ for some fixed λ. For p prime, we set

$$\Omega(p) = \mathrm{image}(-\mathcal{H} \to \mathbb{Z}/p\mathbb{Z})$$

and $\upsilon_{\mathcal{H}}(p) = \#\Omega(p)$. Extend both of these by multiplicativity to squarefree d. We set

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{\upsilon_{\mathcal{H}}(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}.$$

For the GPY method, we set

$$a(n) = \left(\sum_{d \mid (n+h_1)\cdots(n+h_k)} \rho(d) \right)^2$$

for

$$\rho(d) = \mu(d) \left(\frac{\log R/d}{\log R} \right)^{k+\ell} \qquad (d \leq R) \tag{1}$$

with $R \leq x^{1/2}/(\log x)^C$, where C is a constant depending on k, ℓ to be specified later. It will be more convenient to renormalize

$$\rho'(d) = \frac{(\log R)^{k+\ell}}{(k+\ell)!}\rho(d) = \mu(d)\frac{1}{(k+\ell)!}(\log R/d)^{k+\ell} \qquad (d \leq R).$$

2 The main calculation

Lemma 1. *With notation as above, there exist $c > 0$ depending on k, ℓ, such that for C sufficiently large (depending on k, ℓ),*

$$\sum_{x < n \leq 2x} a(n) = \frac{\mathfrak{S}(\mathcal{H})(k+\ell)!^2}{(k+2\ell)!(\log R)^{2k+2\ell}} \binom{2\ell}{\ell} x (\log R)^{k+2\ell} + O\left(\frac{x (\log x)^{k+2\ell-1} (\log\log x)^c}{(\log R)^{2k+2\ell}} \right).$$

Proof. Expanding the square in the definition of $a(n)$, we get the sum over d_1, d_2 of $\rho(d_1)\rho(d_2)$ times the number of $x < n \leq 2x$ with $n \in \Omega(d_1), \Omega(d_2)$. That gives

$$\sum_{x<n\leq 2x} a(n) = x\frac{(k+\ell)!^2}{(\log R)^{2k+2\ell}}\mathcal{T} + O\left(\left(\sum_d |v_{\mathcal{H}}(d)\rho(d)|\right)^2\right)$$

$$\mathcal{T} = \sum_{d_1,d_2} \frac{v_{\mathcal{H}}([d_1,d_2])}{[d_1,d_2]}\rho'(d_1)\rho'(d_2)$$

since $|\Omega(d)| \leq \tau_k(d)$, we can replace the error term with $O(R^2(\log R)^c)$.

We now convert over to a problem in complex analysis, as in the first section of the course. The key formula is

$$\rho'(d) = \frac{\mu(d)}{2\pi i}\int_{(1)} (R/d)^s \frac{ds}{s^{k+\ell+1}},$$

where (α) denotes the vertical contour $\alpha - i\infty \to \alpha + i\infty$. This gives

$$\mathcal{T} = \frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)} F(s_1,s_2,\mathcal{H})\frac{R^{s_1+s_2}}{(s_1 s_2)^{k+\ell+1}}ds_1\, ds_2,$$

with

$$\begin{aligned} F(s_1,s_2,\mathcal{H}) &= \sum_{d_1,d_2} \mu(d_1)\mu(d_2)\frac{v_{\mathcal{H}}([d_1,d_2])}{[d_1,d_2]d_1^{s_1}d_2^{s_2}} \\ &= \prod_p \left(1 - \frac{v_{\mathcal{H}}(p)}{p}(p^{-s_1} + p^{-s_2} - p^{-s_1-s_2})\right) \end{aligned}$$

in the region of absolute convergence.

Now put

$$G(s_1,s_2,\mathcal{H}) = F(s_1,s_2,\mathcal{H})\left(\frac{\zeta(s_1+1)\zeta(s_2+1)}{\zeta(s_1+s_2+1)}\right)^k.$$

Since $v_{\mathcal{H}}(p) = k$ for almost all p, this function is holomorphic and bounded for $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > -c$. In particular, we recover the singular series as

$$\mathfrak{S}(\mathcal{H}) = G(0,0,\mathcal{H}).$$

Now note that from the Euler product expansion, we see that for $\min\{\mathrm{Re}(s_1), \mathrm{Re}(s_2), 0\} = \sigma \geq -c$, we have

$$G(s_1,s_2,\mathcal{H}) = O(\exp(c(\log x)^{-2\sigma}\log\log\log x)). \tag{2}$$

(More specifically, we can uniformly bound the Euler products over $p \leq k^2$ and $p > H$; we get the quoted estimate from the range $k^2 < p \leq H$.)

We use (2) to truncate the infinite integral, but first we shift the contours. Put $U = \exp(\sqrt{\log x})$. We shift the s_1-contour to $L_1 = (\log U)^{-1} + it$, and the s_2-contour to $L_2 = (2\log U)^{-1} + it$. If we now truncate to $|t| \le U$ and $|t| \le U/2$, respectively, we have

$$\mathcal{T} = \frac{1}{(2\pi i)^2} \int_{L_2} \int_{L_1} G(s_1, s_2, \mathcal{H}) \left(\frac{\zeta(s_1+s_2+1)}{\zeta(s_1+1)\zeta(s_2+1)} \right)^k \frac{R^{s_1+s_2}}{(s_1 s_2)^{k+\ell+1}} ds_1\, ds_2 \\ + O(\exp(-c\sqrt{\log x})).$$

We now shift the s_1-contour again, this time to $L_1' = -(\log U)^{-1} + it$ with $|t| \le U$; we pick up residues at $s_1 = 0$ and $s_1 = -s_2$. Again using (2), we get

$$\mathcal{T} = \frac{1}{2\pi i} \int_{L_2} (\mathrm{Res}_{s_1=0} + \mathrm{Res}_{s_1=-s_2})\; ds_2 + O(\exp(-c\sqrt{\log x})).$$

We wish to show that the residue at $s_1 = -s_2$ may be neglected, by rewriting it in terms of the integral over the circle $|s_1+s_2| = (\log x)^{-1}$. In this integral, $G(s_1, s_2, \Omega) = O((\log\log x)^c)$, $R^{s_1+s_2} = O(1)$, $\zeta(s_1+s_2+1) = O(\log x)$. We also have

$$(s_1\zeta(s_1+1))^{-1} = O((|s_2|+1)^{-1}\log(|s_2|+2)).$$

Putting this together,

$$\mathrm{Res}_{s_1=-s_2} \le O\left((\log x)^{k-1} (\log\log x)^c \left(\frac{\log(|s_2|+2)}{|s_2|+1} \right)^{2k} |s_2|^{-2\ell-2} \right),$$

so

$$\mathcal{T} = \frac{1}{2\pi i} \int_{L_2} \mathrm{Res}_{s_1=0}\; ds_2 + O((\log x)^{k+\ell}). \tag{3}$$

It remains to deal with $\mathrm{Res}_{s_1=0}$; note that the pole has order $\ell+1$. If I put

$$Z(s_1, s_2, \mathcal{H}) = G(s_1, s_2, \mathcal{H}) \left(\frac{(s_1+s_2)\zeta(s_1+s_2+1)}{s_1\zeta(s_1+1)s_2\zeta(s_2+1)} \right)^k,$$

then $Z(s_1, s_2, \mathcal{H})$ is holomorphic near $(0,0)$, and

$$\mathrm{Res}_{s_1=0} = \frac{R^{s_2}}{\ell!\, s_2^{\ell+1}} \left(\frac{\partial}{\partial s_1} \right)^{\ell}_{s_1=0} \left(\frac{Z(s_1, s_2, \mathcal{H})}{(s_1+s_2)^k} R^{s_1} \right).$$

We now stuff this into (3) and repeat the operation: that is, we shift the s_2-contour to $L_2' : -(2\log U)^{-1} + it$ for $|t| \le U/2$. Again, the new integral is $O(\exp(-c\sqrt{\log x}))$, so all that is left is the residue at $s_2 = 0$. In other words,

$$\mathcal{T} = \mathrm{Res}_{s_2=0}\,\mathrm{Res}_{s_1=0} + O((\log N)^{k+\ell}).$$

This constitutes success: we have isolated the integral at the point $(0,0)$, so now we will have no trouble evaluating it.

Fix some $\rho > 0$ small, let C_1 be the circle $|s_1| = \rho$, and let C_2 be the circle $|s_2| = 2\rho$. Then

$$\mathcal{T} = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} \frac{Z(s_1, s_2, \mathcal{H}) R^{s_1+s_2}}{(s_1+s_2)^k (s_1 s_2)^{\ell+1}} ds_1\, ds_2 + O((\log x)^{k+\ell}).$$

We now change variables to s, ξ where $s_1 = s$ and $s_2 = s\xi$, over the contours $C : |s| = \rho$ and $C' : |\xi| = 2$. By the same argument as in the runup to (3) (applied to s), this gives

$$\mathcal{T} = \frac{Z(0,0)}{2\pi i (k+2\ell)!} (\log R)^{k+2\ell} \int_{C'} \frac{(\xi+1)^{2\ell}}{\xi^{\ell+1}}\, d\xi + O((\log x)^{k+2\ell-1} (\log\log x)^c).$$

We can now read off the residue of the integrand as $\binom{2\ell}{\ell}$, completing the proof. □

3 Twisting with primes

The second estimate proceeds mostly the same way, so I will skip most details. Note that the translation trick from last time means we don't have to worry about case (b): if $h \in \mathcal{H}$ and $n + h$ is prime, then $a(n, \mathcal{H}) = a(n, \mathcal{H}, h)$.

Lemma 2. *With notation as above, there exist $c > 0$ depending on k, ℓ, such that for C sufficiently large (depending on k, ℓ), we have the following.*

(a) For $h \notin \mathcal{H}$, the quantity

$$\frac{x}{\log x} \sum_{d_1, d_2 \le R} \rho(d_1) \rho(d_2) \frac{g([d_1, d_2])}{\phi([d_1, d_2])}, \tag{4}$$

where g is the multiplicative function with $g(p) = v_{\mathcal{H}}(p) - 1$, equals

$$\frac{\mathfrak{S}(\mathcal{H}, h)}{(\log R)^{2k+2\ell}} \frac{(k+\ell)!^2}{(k+2\ell)!} \binom{2\ell}{\ell} \frac{x}{\log x} (\log R)^{k+2\ell} + O\left(\frac{x (\log x)^{k+2\ell-2} (\log\log x)^c}{(\log R)^{2k+2\ell}} \right).$$

(b) For $h \in \mathcal{H}$, (4) equals

$$\frac{\mathfrak{S}(\mathcal{H})}{(\log R)^{2k+2\ell}} \frac{(k+\ell)!^2}{(k+2\ell+1)!} \binom{2(\ell+1)}{\ell+1} \frac{x}{\log x} (\log R)^{k+2\ell+1} + O\left(\frac{x (\log x)^{k+2\ell-1} (\log\log x)^c}{(\log R)^{2k+2\ell}} \right).$$

Proof. It is a bit more convenient to multiply both sides by $\log x$, pull $\log x$ into the summand, then replace it by $\Lambda(n)$; by the prime number theorem (and the fact that I'm working in a dyadic range), this does not affect the outcome.

In a similar fashion as above, we end up dealing with the exrpession

$$\mathcal{T}' = \frac{1}{(2\pi i)^2} \int_{(1)} \int_{(1)} \prod_p \left(1 - \frac{v_{\mathcal{H},h}(p) - 1}{p-1} (p^{-s_1} + p^{-s_2} - p^{-s_1-s_2}) \right) \frac{R^{s_1+s_2}}{(s_1 s_2)^{k+\ell+1}}\, ds_1\, ds_2.$$

Everything proceeds as before *unless* $v_{\mathcal{H},h}(p) = p$ for some p. In that case, the Euler product above vanishes at one of $s_1 = 0$ or $s_2 = 0$, to order equal to the number of primes for which $v_{\mathcal{H},h}(p) = 0$. But this can only happen for $p \le k+1$, and so we can still proceed as above: all that changes is that now the main term vanishes, consistent with $\mathfrak{S}(\mathcal{H} \cup \{h\}) = 0$. □