Here are some suggestions about how to apply the Selberg sieve; this should help with some of the exercises on the previous handout (the bound on twin primes, and the Brun-Titchmarsh inequality).

1 Review of the setup

Recall the setup.

**Theorem 1** (Selberg). Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be an arithmetic function with finite support. Let $P$ be a set of primes, and put $P(z) = \prod_{p \leq z, p \in P} p$. For $d | P(z)$, write

$$A_d = \sum_{n \equiv 0 (d)} f(n) = g(d)X + r_d(z)$$

for $X > 0$ and $g$ a multiplicative function with $0 < g(p) < 1$ for all $p \in P$. Let $h(d)$ be a multiplicative function with $h(p) = g(p)(1 - g(p))^{-1}$ for all $p \in P$, and put

$$H = \sum_{d < \sqrt{y}d | P(z)} h(d)$$

for some $y > 1$. Then

$$S(z) = \sum_{(n,P(z))=1} f(n) \leq XH^{-1} + \sum_{d | P(z)} \lambda^+(d)r_d(z), \quad (1)$$

for

$$\lambda^+(n) = \sum_{d_1, d_2 : \text{lcm}(d_1, d_2) = n} \rho(d_1)\rho(d_2)$$

$$\rho(d) = \mu(d) \frac{h(d)}{g(d)}H^{-1} \sum_{n < \sqrt{y} : \gcd(d, n) = 1} h(n).$$

Also recall that we could bound $\lambda^+(d)$ by $\tau_3(d)$, the number of ways to write $d$ as a product of 3 positive integers.

2 Interlude: bounding sums of multiplicative functions

Let $f$ be a multiplicative function, for which we want to bound $\sum_{n \leq x} f(n)$. Here is an argument that does this for us (due to Wirsing), assuming some control over the values of $f$ at prime powers.
To be specific, let $e$ be the arithmetic function defined by the following identity of formal Dirichlet series:

$$
\sum_{n=1}^{\infty} e(n)n^{-s} = -\frac{d}{ds}\log \sum_{n=1}^{\infty} f(n)n^{-s}.
$$

We will impose the condition that for some $\kappa > 0$,

$$
\sum_{n \leq x} e(n) = \kappa \log x + O(1) \tag{2}
$$

and

$$
\sum_{n \leq x} |f(n)| = O(\log |\kappa| x). \tag{3}
$$

(The superfluous absolute value in (3) is included because it actually suffices to take $\kappa > -1/2$, but we won’t use this.)

Define

$$
M_f(x) = \sum_{n \leq x} f(n),
$$

which is what we want to estimate. We first obtain

$$
(\kappa + 1) \sum_{n \leq x} f(n) \log n = \kappa M_f(x) \log x + O(\log^x x) \tag{4}
$$

(exercise). Since

$$
\sum_{n \leq x} f(n) \log(x/n) = \int_1^x M_f(y)y^{-1} \, dy,
$$

we obtain

$$
\Delta(x) = M_f(x) \log x - (\kappa + 1) \int_2^x M_f(y)y^{-1} \, dy = O(\log^x x).
$$

We next derive the following identity:

$$
M_f(x) = \log^x x \int_2^x -\Delta(y)d(\log y)^{-\kappa-1} + \Delta(x) \log^{-1} x \tag{5}
$$

(exercise). This implies

$$
M_f(x) = c_f \log^x x + O(\log^{x-1} x)
$$

for

$$
c_f = -\int_2^\infty \Delta(y)d(\log y)^{-\kappa-1},
$$

but it would be nice to be able to describe $c_f$ more explicitly. Fortunately this is possible: we have

$$
c_f = \frac{1}{\Gamma(\kappa + 1)} \prod_p (1 - p^{-1})^x (1 + f(p) + f(p^2) + \cdots) \tag{6}
$$

(exercise).
3 Bounding the main term

To get an upper bound on the main term $XH^{-1}$, we need a lower bound on $H$. A simple example occurs when $g(d) = d^{-1}$; see exercises.

A more generic example occurs when we have

$$\sum_{p \leq x} g(p) \log p = \kappa \log x + O(1)$$

for some $\kappa > 0$, and

$$\sum_p g(p)^2 \log p < \infty.$$ 

For instance, this holds if $g(p) = c/p$. By Wirsing’s bound, we get

$$H = c \log^\kappa \sqrt{y(1 + O(\log^{-1} y))}$$

$$c = \frac{1}{\Gamma(\kappa + 1)} \prod_p (1 - g(p))^{-1}(1 - p^{-1})^\kappa.$$ 

This can be more usefully written as

$$H^{-1} = 2^\kappa \Gamma(\kappa + 1) H_g \log^{-\kappa} y(1 + O(\log^{-1} y)), \quad (7)$$

where

$$H_g = \prod_p (1 - g(p))(1 - p^{-1})^{-\kappa}.$$ 

4 Bounding the error term

Suppose our function $g$ satisfies the conditions

$$g(d)d \geq 1 \quad (d|P(z)) \quad (8)$$

and

$$\sum_{y \leq p \leq x} g(p) \log p = O(\log(2x/y)). \quad (9)$$

Suppose also that the individual error terms $r_d$ are not too large:

$$|r_d(z)| \leq g(d)d \quad (d|P(z)). \quad (10)$$

Then it is straightforward to derive the bound

$$\left| \sum_{d|P(z)} \lambda^+(d) r_d(z) \right| \leq y \log^{-2} y \quad (11)$$

(exercise).
Exercises

1. In the Selberg sieve, prove that
   \[ H > \log \sqrt{y}. \]
   Moreover, if we instead take \( g(d) = d^{-1} \) and \( P \) to be the set of all primes, then
   \[ H > (\log \sqrt{y}) \prod_{p \mid q} (1 - g(p)). \]

2. Prove (4).

3. Prove (5).

4. Prove (6). (Hint: write \( \sum_{n=1}^{\infty} f(n)n^{-s} \) in terms of \( c_f \) by partial summation, then multiply by \( \zeta(s+1)^s \) and compare to the Euler product.)

5. Prove (11). (Hint: first bound the sum on the left by
   \[ \left( \sum_{d < \sqrt{y}} |\rho_d| g(d) d \right)^2 \leq \left( \frac{1}{H} \sum_{n < \sqrt{y}} h(n) \sigma(n) \right)^2, \]
   where \( \sigma \) is the usual sum-of-divisors function. Then apply the prime number theorem plus partial summation to control this.)