18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
Applying the Selberg sieve

Here are some suggestions about how to apply the Selberg sieve; this should help with some of the exercises on the previous handout (the bound on twin primes, and the Brun-Titchmarsh inequality).

1 Review of the setup

Recall the setup.

Theorem 1 (Selberg)**.** *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be an arithmetic function with finite support. Let P be a set of primes, and put $P(z) = \prod_{p \leq z, p \in P} p$. For $d | P(z)$, write*

$$A_d = \sum_{n \equiv 0\,(d)} f(n) = g(d)X + r_d(z)$$

for $X > 0$ and g a multiplicative function with $0 < g(p) < 1$ for all $p \in P$. Let $h(d)$ be a multiplicative function with $h(p) = g(p)(1 - g(p))^{-1}$ for all $p \in P$, and put

$$H = \sum_{d < \sqrt{y}, d | P(z)} h(d)$$

for some $y > 1$. Then

$$S(z) = \sum_{(n, P(z)) = 1} f(n) \leq XH^{-1} + \sum_{d | P(z)} \lambda^+(d) r_d(z), \tag{1}$$

for

$$\lambda^+(n) = \sum_{d_1, d_2 : \mathrm{lcm}(d_1, d_2) = n} \rho(d_1)\rho(d_2)$$
$$\rho(d) = \mu(d) \frac{h(d)}{g(d)} H^{-1} \sum_{n < \sqrt{y}/d : \gcd(d, n) = 1} h(n).$$

Also recall that we could bound $\lambda^+(d)$ by $\tau_3(d)$, the number of ways to write d as a product of 3 positive integers.

2 Interlude: bounding sums of multiplicative functions

Let f be a multiplicative function, for which we want to bound $\sum_{n \leq x} f(n)$. Here is an argument that does this for us (due to Wirsing), assuming some control over the values of f at prime powers.

To be specific, let e be the arithmetic function defined by the following identity of formal Dirichlet series:

$$\sum_{n=1}^{\infty} e(n)n^{-s} = -\frac{d}{ds}\log\sum_{n=1}^{\infty} f(n)n^{-s}.$$

We will impose the condition that for some $\kappa > 0$,

$$\sum_{n\le x} e(n) = \kappa\log x + O(1) \tag{2}$$

and

$$\sum_{n\le x} |f(n)| = O(\log^{|\kappa|} x). \tag{3}$$

(The superfluous absolute value in (3) is included because it actually suffices to take $\kappa > -1/2$, but we won't use this.)

Define

$$M_f(x) = \sum_{n\le x} f(n),$$

which is what we want to estimate. We first obtain

$$(\kappa+1)\sum_{n\le x} f(n)\log n = \kappa M_f(x)\log x + O(\log^{\kappa} x) \tag{4}$$

(exercise). Since

$$\sum_{n\le x} f(n)\log(x/n) = \int_1^x M_f(y)y^{-1}\,dy,$$

we obtain

$$\Delta(x) = M_f(x)\log x - (\kappa+1)\int_2^x M_f(y)y^{-1}\,dy = O(\log^{\kappa} x).$$

We next derive the following identity:

$$M_f(x) = \log^{\kappa} x\int_2^x -\Delta(y)d(\log y)^{-\kappa-1} + \Delta(x)\log^{-1} x \tag{5}$$

(exercise). This implies

$$M_f(x) = c_f\log^{\kappa} x + O(\log^{\kappa-1} x)$$

for

$$c_f = -\int_2^{\infty}\Delta(y)d(\log y)^{-\kappa-1},$$

but it would be nice to be able to describe c_f more explicitly. Fortunately this is possible: we have

$$c_f = \frac{1}{\Gamma(\kappa+1)}\prod_p (1-p^{-1})^{\kappa}(1+f(p)+f(p^2)+\cdots) \tag{6}$$

(exercise).

3 Bounding the main term

To get an upper bound on the main term XH^{-1}, we need a lower bound on H. A simple example occurs when $g(d) = d^{-1}$; see exercises.

A more generic example occurs when we have

$$\sum_{p \leq x} g(p) \log p = \kappa \log x + O(1)$$

for some $\kappa > 0$, and

$$\sum_p g(p)^2 \log p < \infty.$$

For instance, this holds if $g(p) = c/p$. By Wirsing's bound, we get

$$H = c \log^\kappa \sqrt{y}(1 + O(\log^{-1} y))$$
$$c = \frac{1}{\Gamma(\kappa+1)} \prod_p (1 - g(p))^{-1}(1 - p^{-1})^\kappa.$$

This can be more usefully written as

$$H^{-1} = 2^\kappa \Gamma(\kappa+1) H_g \log^{-\kappa} y(1 + O(\log^{-1} y)), \tag{7}$$

where

$$H_g = \prod_p (1 - g(p))(1 - p^{-1})^{-\kappa}.$$

4 Bounding the error term

Suppose our function g satisfies the conditions

$$g(d)d \geq 1 \qquad (d|P(z)) \tag{8}$$

and

$$\sum_{y \leq p \leq x} g(p) \log p = O(\log(2x/y)). \tag{9}$$

Suppose also that the individual error terms r_d are not too large:

$$|r_d(z)| \leq g(d)d \qquad (d|P(z)). \tag{10}$$

Then it is straightforward to derive the bound

$$\left| \sum_{d|P(z)} \lambda^+(d) r_d(z) \right| \leq y \log^{-2} y \tag{11}$$

(exercise).

Exercises

1. In the Selberg sieve, prove that

$$H > \log \sqrt{y}.$$

Moreover, if we instead take $g(d) = d^{-1}$ and P to be the set of all primes, then

$$H > (\log \sqrt{y}) \prod_{p|q} (1 - g(p)).$$

2. Prove (4).

3. Prove (5).

4. Prove (6). (Hint: write $\sum_{n=1}^{\infty} f(n) n^{-s}$ in terms of c_f by partial summation, then multiply by $\zeta(s+1)^{\kappa}$ and compare to the Euler product.)

5. Prove (11). (Hint: first bound the sum on the left by

$$\left(\sum_{d < \sqrt{y}} |\rho_d| g(d) d \right)^2 \leq \left(\frac{1}{H} \sum_{n < \sqrt{y}} h(n) \sigma(n) \right)^2,$$

where σ is the usual sum-of-divisors function. Then apply the prime number theorem plus partial summation to control this.)