In this unit, we derive some results about the location of the zeroes of the Riemann zeta function, including a small zero-free region inside the critical strip.

1 Order of an entire function

For $\alpha > 0$, an entire function $f : \mathbb{C} \to \mathbb{C}$ is said to have order $\leq \alpha$ if for all $\beta > \alpha$,

$$f(z) = O(\exp |z|^\beta) \quad (|z| \to \infty).$$

We say $f$ has order $\alpha$ if it has order $\leq \alpha$ but not order $\leq \beta$ for any $\beta < \alpha$.

Lemma 1. The function

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

satisfies

$$|\xi(s)| < \exp(C|s| \log |s|) \quad (|s| \to \infty),$$

and so is of order $\leq 1$. (An analogue is true for $L$-functions, but that is too easy even to give as an exercise.)

Proof. By the functional equation $\xi(s) = \xi(1-s)$, it suffices to check for $|\text{Re}(s)| \geq 1/2$, in which case

$$\left|\frac{1}{2} s(s-1)\pi^{-s/2}\right| < \exp(C_1|s|)$$

$$|\Gamma(s/2)| < \exp(C_2|s| \log |s|)$$

(see exercises for the second estimate). For $\zeta$, we use the integral representation from the first lecture:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty (x - |x|) x^{-s-1} \, dx \quad (\text{Re}(s) > 0).$$

For $\text{Re}(s) \geq 1/2$, the integral is bounded, so $|\zeta(s)| < C_3|s|$. This yields the claim. \qed

There is a rich theory of integral functions of finite order due to Hadamard (which I believe was introduced originally for the very purpose of studying $\zeta$). The basic idea is to generalize the fact that a polynomial can be written as a product of linear factors (the Fundamental Theorem of Algebra), to write an entire function as a product of one factor for each zero times an exponential.

To do this, one must first control the number of zeroes of $f$ in a disc. There is no harm in assuming that $f(0) \neq 0$, since otherwise we just divide by a suitable power of $z$. Then recall the following fact from complex analysis.
Theorem 2 (Jensen’s formula). If \( f(0) \neq 0 \) and \( f \) has no zeroes on the circle \( |z| = R \), then
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\theta})| \, d\theta = \log |f(0)| + \sum_{\rho} (\log R - \log |\rho|),
\]
where \( \rho \) runs over the zeroes of \( f \) in the disc \( |z| < R \) counted with multiplicity.

Proof. Write \( f(z) = (z - \rho_1) \cdots (z - \rho_n)g(z) \), where \( g \) is nonzero on the disc \( |z| \leq R \), and check the equality for each factor individually. For \( z - \rho_i \), this is an easy exercise; for \( g \), apply the Cauchy residue formula to the contour integral \( \int \log(g(z)) \frac{dz}{z} \) around the circle \( |z| = R \), then take real parts.

The right side is also
\[
\log |f(0)| + \int_{0}^{R} \frac{\#\{\rho : |\rho| < r\} \, dr}{r}
\]
If \( \log |f(z)| < r(|z|) \) for some function \( r \), then the left side of Jensen’s formula is bounded by \( 2r(R) \), whereas the right side is at least
\[
\log |f(0)| + \log(2) \#\{\rho : |\rho| \leq R/2\}.
\]
Consequently, if \( r(R) = O(R^\alpha) \), then the number of zeroes of \( f \) in the disc \( |\rho| \leq R \) is also \( O(R^\alpha) \). Similarly, the fact that \( \log |\xi(s)| = O(|s| \log |s|) \) implies that the number of zeroes of \( \xi \) with \( |\text{Im}(s)| \leq T \) is \( O(T \log T) \), which I claimed without proof in the previous unit.

Now let \( f \) be entire of order \( \leq 1 \). Let \( \rho_1, \rho_2, \ldots \) be the zeroes of \( f \) sorted so that \( |\rho_1| \leq |\rho_2| \leq \cdots \), and put
\[
h(z) = \prod_{n=1}^{\infty} (1 - z/\rho_n)e^{z/\rho_n}
\]
Note that this converges uniformly on any disc, because the multiplicand is
\[
1 + \frac{1}{2} \left( \frac{z}{\rho_n} \right)^2 + O \left( \left( \frac{z}{\rho_n} \right)^3 \right)
\]
and the fact that the number of roots of norm \( \leq R \) is \( O(R^{1+\epsilon}) \) implies that \( \sum 1/\rho_n^2 \) converges (by partial summation). By a somewhat intricate argument (see Davenport §11 or Ahlfors), it can be shown that \( f/h \) is also of order \( \leq 1 \). Since \( f/h \) has no zeroes, the function \( g(z) = \log(f(z)/h(z)) \) is entire and satisfies \( |g(z)| = O(|z|^{1+\epsilon}) \). Consequently,
\[
g_2(z) = \frac{g(z) - g(0) - g'(0)z}{z^2}
\]
is entire and bounded, hence constant by Liouville’s theorem. This yields the following.

Theorem 3 (Hadamard). Let \( f(z) \) be an entire function of order \( \leq 1 \). Then
\[
f(z) = e^{A + Bz} \prod_{n=1}^{\infty} (1 - z/\rho_n)e^{z/\rho_n}
\]
for some constants \( A, B \).
2 A zero-free region for \( \zeta \)

We now use the product representation for \( \xi \) to obtain a zero-free region for \( \zeta \). The idea (due to de la Vallée Poussin (1899)) is to squeeze a bit of extra information out of the proof we used for nonvanishing on the line \( \text{Re}(s) = 1 \). One way to phrase that argument: since

\[
\text{Re}(\log(\zeta(s))) = \sum_p \sum_{n=1}^{\infty} \frac{1}{n} \cos(\text{Im}(s) \log p^n) p^{-n \text{Re}(s)}
\]

and

\[
3 + 4 \cos \theta + \cos 2\theta \geq 0,
\]

we have

\[
3 \text{Re}(\log \zeta(\sigma)) + 4 \text{Re}(\log \zeta(\sigma + it)) + \text{Re}(\log \zeta(\sigma + 2it)) \geq 0 \quad (\sigma > 1, t \in \mathbb{R})
\]

whereas if \( \zeta(1 + it) \) vanished, then the sum would tend to \(-\infty\) as \( \sigma \to 1^{+} \) (because \( 4 > 3 \)).

We can apply the same argument with \( \log \zeta \) replaced by its negative derivative

\[
- \text{Re} \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-\text{Re}(s)} \cos(\text{Im}(s) \log n)
\]

to obtain an analogous inequality

\[
-3 \text{Re} \frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4 \text{Re} \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} - \text{Re} \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \geq 0 \quad (\sigma > 1, t \in \mathbb{R}). \quad (1)
\]

Let’s see how to use (1) to get some information about zeroes just past the line \( \text{Re}(s) = 1 \). We do this by bounding above each term on the left side of (1) for \( \sigma \) slightly bigger than 1. For starters, since \( \zeta \) has a simple pole at \( s = 1 \),

\[
- \frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + *
\]

where every * in this argument is a positive constant, but no two need be the same.

Applying Hadamard’s theorem and taking a logarithmic derivative, we get

\[
\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).
\]

Adjusting to get rid of the gamma factors, we get

\[
- \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s - 1} - B - \frac{1}{2} \log \pi + \frac{1}{2} \Gamma'(s + 1/2) \Gamma((s + 1)/2) - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).
\]
For $1 \leq \text{Re}(s) \leq 2$ and $|\text{Im}(s)| \geq 1$, everything on the right side aside from the sum over $\rho$ is dominated by $\ast \log |\text{Im}(s)|$. Hence taking real parts, we obtain

$$-\text{Re} \frac{\zeta'(s)}{\zeta(s)} < \ast \log |\text{Im}(s)| - \sum_{\rho} \text{Re} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

Since $\text{Re}(\rho) > 0$ and $\text{Re}(s - \rho) > 0$, we also have $\text{Re}(1/\rho) > 0$ and $\text{Re}(1/(s - \rho)) > 0$, so the sum over $\rho$ is positive. Hence

$$-\text{Re} \frac{\zeta'(s)}{\zeta(s)} < \ast \log |\text{Im}(s)|;$$

this is the estimate I’ll use for $s = \sigma + 2it$.

Let $t$ be the imaginary part of a zero $\rho$ of $\zeta$; I will bound $-\text{Re} \frac{\zeta'(s)}{\zeta(s)}$ for $s = \sigma + it$ by keeping only the summand corresponding to $\rho$. Namely, if $\rho = \beta + it$, then I get

$$-\text{Re} \frac{\zeta'(s)}{\zeta(s)} < \ast \log |t| - \frac{1}{\sigma - \beta}. $$

From (1), I now deduce

$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + \ast \log |t|. $$

For $\sigma = 1 + \ast/(\log |t|)$, I can deduce

$$\beta < 1 - \frac{\ast}{\log |t|}. $$

In other words:

**Theorem 4.** There exists a constant $c > 0$ such that there is no zero of $\zeta$ in the region $\text{Re}(s) \geq 1 - c/\log \text{Im}(s), \text{Im}(s) \geq 1$.

By von Mangoldt’s formula (presented in the previous unit, with proof still to follow), this yields a nontrivial error bound in the prime number theorem, namely

$$\pi(x) = \text{li}(x) + O(x \exp(-c\sqrt{\log x})).$$

(exercise).

## 3 What about $L$-functions?

The previous argument goes through more or less unchanged for $L$-functions. But there is a new complication: remember that we only looked at zeroes whose imaginary part was not too small. We took $|\text{Im}(s)| \geq 1$, but the lower bound could have been any fixed positive constant.

The real issue is that while we can check once and for all that $\zeta(s)$ has no zeroes on the real line, we cannot rule this out for $L$-functions. But $L(s, \chi)$ could in principle have a real zero; such a hypothetical zero is called a *Siegel zero*. These can only occur for real nonprincipal characters.
Exercises

1. Prove that $1/\Gamma$ is entire of order $\leq 1$. Then prove that

$$ \frac{1}{s\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} (1 + s/n) e^{-s/n} \quad (s \neq 0, -1, -2, \ldots), $$

where $\gamma$ is Euler's constant, by applying Hadamard's theorem.

2. Prove that

$$ \frac{\Gamma'(s)}{\Gamma(s)} = \log(s) + O(|s|^{-1}) \quad (|s| \to \infty, \text{Re}(s) \geq 1/2). $$

(Hint: use the previous exercise.)

3. Derive the estimate

$$ |\Gamma(s/2)| < \exp(C_2|s| \log |s|) \quad (\text{Re}(s) \geq 1/2) $$

by first proving a suitably strong version of Stirling's formula, e.g.,

$$ \log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}) \quad (|s| \to \infty, \text{Re}(s) \geq 1/2). $$

4. Prove that a function of order $\leq \alpha$ need not satisfy $|f(z)| = O(\exp(|z|^\alpha))$. (Hint: look at $\zeta$ on the positive real axis.)

5. Find the constants $A$ and $B$ in the product representation for $\xi$ given by Hadamard's theorem. Then deduce as a corollary that $\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi$.

6. Use the zero-free region and von Mangoldt's formula to prove that for some $c > 0$,

$$ \pi(x) = \text{li}(x) + O(x \exp(-c\sqrt{\log x})). $$

(By contrast, the leading term is $x \exp(-\log \log x)$.)