18.786: Topics in Algebraic Number Theory (spring 2006)
Problem Set 9, due Thursday, April 27

1. Janusz p. 118, exercise 2.

2. Janusz p. 118, exercise 3.

3. Janusz p. 118, exercise 4. Optional (not to be turned in): the other exercises in that section.

4. Let K be a finite extension of $\mathbb{Q}_p$. Let $f(x) = x^n + \sum_{i=0}^{n-1} f_i x^i$ be a monic polynomial of degree n over K, which factors completely over K with distinct roots $r_1, \dots, r_n$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $g(x) = x^n + \sum_{i=0}^{n-1} g_i x^i$ is a monic polynomial of degree n such that $|f_i - g_i| < \delta$, then g has n roots $s_1, \dots, s_n$ in K, which can be labeled so that $|r_i - s_i| < \epsilon$ for $i = 1, \dots, n$. That is, the roots of f vary continuously with the coefficients. (If the roots are not distinct, the roots of g may only lie in an extension of K, but otherwise the conclusion still holds.)

5. Let K be a finite extension of $\mathbb{Q}_p$. Prove *Krasner's Lemma*: if $\alpha_1, \dots, \alpha_n \in \overline{K}$ are conjugates, and $\beta \in \overline{K}$ satisfies
$$|\alpha_1 - \beta| < |\alpha_1 - \alpha_i| \quad (i = 2, \dots, n),$$
then $K(\alpha_1) \subseteq K(\beta)$.

6. (Abhyankar's Lemma) Let K be a finite extension of $\mathbb{Q}_p$. A finite extension L/K is said to be *tamely ramified* if $e(\mathfrak{m}_L/\mathfrak{m}_K)$ is coprime to p. Let L_1, L_2 be tamely ramified extensions of K such that $e(\mathfrak{m}_{L_1}/\mathfrak{m}_K)$ divides $e(\mathfrak{m}_{L_2}/\mathfrak{m}_K)$. Prove that the compositum L_1L_2 is unramified over L_2. (Hint: it is safe to check this after making an unramified extension of K, so you can assume L_1 and L_2 are both Kummer extensions.)

7. (Dwork) Let p be a prime number. Show that $\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(\pi)$ for π a $(p-1)$-st root of $-p$. (Hint: either of the previous two exercises might be helpful, or you can explicitly construct a series in π converging to ζ_p.)

8. (Optional because it uses some topology, but strongly recommended) Let K be a number field. Let $\mathbb{A}_K$ be the subring of the product $\prod_v K_v$, where v runs over places and K_v is the completion at v, consisting of tuples (a_v) in which $a_v \in \mathfrak{o}_{K_v}$ for all but finitely many finite places v (no condition is imposed at infinite places). Give $\mathbb{A}_K$ the topology with a basis of open sets given by products $\prod_v U_v$, with U_v open in K_v and $U_v = \mathfrak{o}_{K_v}$ for all but finitely many finite v. Prove that K, which naturally embeds into $\mathbb{A}_K$ via the maps $K \hookrightarrow K_v$, is a *discrete* subgroup of $\mathbb{A}_K$ and that the quotient $\mathbb{A}_K/K$ is *compact*; that is, in some sense K is a "full lattice" in $\mathbb{A}_K$. (Hint: start with Tykhonov's theorem that any product of compact spaces is compact.) The ring $\mathbb{A}_K$ is the *ring of adèles* of K; we'll likely see it again later. (There's a multiplicative analogue too; more on that later.)