In this unit, we introduce $p$-adic closed discs and annuli, but in a purely ring-theoretic fashion. This avoids having to introduce any $p$-adic analytic geometry.

Throughout the unit (and in all later units, unless explicitly contravened), let $K$ be a field complete for a nontrivial nonarchimedean valuation $| \cdot |$. Assume that $K$ has characteristic 0, but the residue field $\kappa_K$ has characteristic $p > 0$. Also assume that things are normalized so that $|p| = p^{-1}$.

1 Power series on closed discs and annuli

We start by introducing some rings that should be thought of as the analytic functions on a closed disc $|t| \leq \beta$, or a closed annulus $\alpha \leq |t| \leq \beta$. As noted in the introduction, this is more properly done in a framework of $p$-adic analytic geometry, but we will avoid this framework.

For $\alpha, \beta > 0$, put

$$K(\alpha/t, t/\beta) = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i \in K[[t, t^{-1}]] : \lim_{i \rightarrow \pm \infty} |c_i| \rho^i = 0 \ (\rho \in [\alpha, \beta]). \right\}.$$  

That is, consider formal bidirectional power series which converge whenever you plug in a value for $t$ with $|t| \in [\alpha, \beta]$, or in other words, when $\alpha/|t|$ and $|t|/\beta$ are both at most 1; it suffices to check for $\rho = \alpha$ and $\rho = \beta$. Although formal bidirectional power series do not form a ring, the subset $K(\alpha/t, t/\beta)$ does form a ring under the expected operations.

If $\alpha = 0$, the only reasonable interpretation of the previous definition is to require $c_i = 0$ for $i < 0$. When there are no negative powers of $t$, it is redundant to require the convergence for $\rho < \beta$. In other words,

$$K(0/t, t/\beta) = K(t/\beta) = \left\{ \sum_{i=0}^{\infty} c_i t^i \in K[[t]] : \lim_{i \rightarrow \infty} |c_i| \beta^i = 0 \right\}.$$  

One could also allow $\beta = \infty$ for a similar effect in the other direction. More succinctly put, we identify $K(\alpha/t, t/\beta)$ with $K(\beta^{-1}/t^{-1}, t^{-1}/\alpha^{-1})$.

2 Gauss norms and Newton polygons

The rings $K(\alpha/t, t/\beta)$ quite a lot like polynomial rings (or Laurent polynomial rings, in case $\alpha \neq 0$) in one variable. The next few statements are all instances of this analogy.
From the definition of $K\langle \alpha/t, t/\beta \rangle$, we see that it carries a well-defined $\rho$-Gauss norm

$$\left| \sum_i c_i t^i \right|_{\rho} = \max_i \{|c_i|^{1-\rho}\}$$

for any $\rho \in [\alpha, \beta]$. For $\rho = \alpha = 0$, this reduces to simply $|c_0|$.

The additive version is this is to take $r \in [-\log \beta, -\log \alpha]$ and put

$$v_r \left( \sum c_i t^i \right) = \min_i \{v(c_i) + ri\},$$

where $v(c) = -\log |c|$. This is the same formula as we had for the sloped valuation function on a polynomial ring, so we may repeat the proof to obtain the following.

**Lemma 1.** For $r \in [-\log \beta, -\log \alpha]$, the function $v_r$ on $K\langle \alpha/t, t/\beta \rangle$ is a valuation; in particular, $v_r(xy) = v_r(x) + v_r(y)$. Equivalently, for $\rho \in [\alpha, \beta]$, the $\rho$-Gauss norm on $K\langle \alpha/t, t/\beta \rangle$ is really a norm; that is, it indeed satisfies $|fg|_\rho = |f|_\rho |g|_\rho$.

One may define the **Newton polygon** for an element $x = \sum x_i t^i \in K\langle \alpha/t, t/\beta \rangle$ as the lower convex hull of the set

$$\{(-i, v(x_i)) : i \in \mathbb{Z}, x_i \neq 0\},$$

except that we only keep the slopes in $[-\log \beta, -\log \alpha]$.

**Proposition 2.** Let $x = \sum i x_i t^i \in K\langle \alpha/t, t/\beta \rangle$ be nonzero.

(a) The Newton polygon of $x$ has finite width.

(b) The function $r \mapsto v_r(x)$ on $[-\log \beta, -\log \alpha]$ is continuous, piecewise affine, and convex.

(c) The function $\rho \mapsto |x|_\rho$ on $[\alpha, \beta]$ is continuous and log-concave. The log-concavity means that $\rho, \sigma \in [\alpha, \beta]$ and $c \in [0,1]$, put $\tau = \rho^c \sigma^{1-c}$; then

$$|x|_\tau \leq |x|_\rho^c |x|_\sigma^{1-c}.$$

(d) If $\alpha = 0$, then $v_r$ is decreasing on $[-\log \beta, \infty)$; in other words, for all $\rho \in [0, \beta]$, $|x|_\rho \leq |x|_\beta$.

Part (c) should be thought of as a nonarchimedean analogue of the Hadamard three circle theorem.

**Proof.** We have (a) because there is a least $i$ for which $|c_i|^{1-\rho}$ is maximized, and there is a greatest $j$ for which $|c_j|^{1-\rho}$ is maximized. This implies (b) because as in the polynomial case, we may interpret $v_r(x)$ as the $y$-intercept of the supporting line of the Newton polygon of slope $r$. This in turn implies (c), and (d) is a remark made earlier. \qed
When dealing with the ring $K\langle \alpha/t, t/\beta \rangle$, the following completeness property will be extremely useful.

**Proposition 3.** The ring $K\langle \alpha/t, t/\beta \rangle$ is Fréchet complete for the norms $| \cdot |_\rho$ for all $\rho \in I$. That is, if $\{x_n\}_{n=0}^\infty$ is a sequence which is simultaneously Cauchy under $| \cdot |_\rho$ for all $\rho \in I$, then it is convergent. (By Proposition 2, it suffices to check the Cauchy property at each nonzero endpoint of $I$.)

**Proof.** Exercise. □

For instance, the completeness property is used in the construction of multiplicative inverses.

**Lemma 4.** If $\alpha = 0$ (resp. $\alpha > 0$), a nonzero element $f \in K\langle \alpha/t, t/\beta \rangle$ is a unit if and only if $v_r$ is constant (resp. affine) on $[-\log(\beta), -\log(\alpha)]$.

**Proof.** We will just consider the case $\alpha > 0$; the other case is similar (and easier). Put $f = \sum_{i \in \mathbb{Z}} f_i t^i$. Note that the following are equivalent:

(a) there is a single $i$ for which $|f_\rho| = |f_i|\rho^i$ for all $\rho \in [\alpha, \beta]$;

(b) the function $r \mapsto v_r(f)$ on $[-\log(\beta), -\log(\alpha)]$ is affine;

(c) the Newton polygon of $f$ has no slopes in $[-\log(\beta), -\log(\alpha)]$.

By (c), these conditions all hold if $f$ is a unit. Conversely, if these conditions hold, then the series

$$(f_i t_i)^{-1}(1 - (f_i t_i - f)/(f_i t_i))^{-1} = \sum_{j=0}^\infty (f_i t_i - f)^j (f_i t_i)^{-j-1}$$

converges by Proposition 3, and its limit is an inverse of $f$. □

### 3 Factorization results

**Proposition 5 (Weierstrass preparation).** Suppose that $f = \sum_{i \in \mathbb{Z}} f_i t^i \in K\langle \alpha/t, t/\beta \rangle$, and that $\rho \in [\alpha, \beta]$ is such that there is a unique $m \in \mathbb{Z}$ maximizing $|f_m|\rho^m$. Then there is a unique factorization $f = f_m t^m g h$ with

$$g \in K\langle \alpha/t, t/\beta \rangle \cap K[t] = K(t/\beta),$$

$$h \in K\langle \alpha/t, t/\beta \rangle \cap K[t^{-1}] = K(\alpha/t),$$

$|g|_\rho = |g_0| = 1$, and $|h - 1|_\rho < 1$.

**Proof.** The master slope factorization applies thanks to Property 3. □

In light of the finite width property of the Newton polygon, the following should not be a surprise.
**Proposition 6** (More Weierstrass preparation). For \( f \in K\langle \alpha/t, t/\beta \rangle \), there exists a polynomial \( P \in K[t] \) and a unit \( g \in K\langle \alpha/t, t/\beta \rangle^\times \) such that \( f = P g \). In particular, \( K\langle \alpha/t, t/\beta \rangle \) is a principal ideal domain.

**Proof.** Using Proposition 5, we may reduce to two instances of the case \( \alpha = 0 \), so we restrict to that case hereafter. Put \( f = \sum_i f_i t^i \), and choose \( m \) maximizing \( |f_m|\beta^m \). Let \( R \) be the ring of formal sums \( \sum_i c_i t^i \) of series with \( |c_i|\beta^i \) bounded as \( i \to -\infty \) and tending to 0 as \( i \to +\infty \). Let \( e \) be the inverse of \( \sum_{i=0}^m f_i t^i \) in \( R \), and apply master slope factorization to factor \( ef = gh \) in \( R \), in which \( g \) is a unit in \( K\langle t/\beta \rangle \) by Lemma 4. Now \( h \sum_{i=0}^m f_i t^i = fg^{-1} \) belongs to

\[ K[t] \cap t^m K[t^{-1}] \]

It is thus a polynomial of degree \( m \), proving the claim. \( \square \)

We will make frequent and often implicit use of the following patching lemma.

**Lemma 7** (Patching lemma). Suppose \( \alpha \leq \gamma \leq \beta \leq \delta \). Let \( M_1 \) be a finite free module over \( K\langle \alpha/t, t/\beta \rangle \), let \( M_2 \) be a finite free module over \( K\langle \gamma/t, t/\delta \rangle \), and suppose we are given an isomorphism

\[ \psi : M_1 \otimes K\langle \gamma/t, t/\beta \rangle \cong M_2 \otimes K\langle \gamma/t, t/\beta \rangle. \]

Then we can find a finite free module \( M \) over \( K\langle \alpha/t, t/\delta \rangle \) and isomorphisms \( M_1 \cong M \otimes K\langle \alpha/t, t/\beta \rangle \), \( M_2 \cong M \otimes K\langle \gamma/t, t/\delta \rangle \) inducing \( \psi \). Moreover, \( M \) is determined by this requirement up to unique isomorphism.

**Proof.** We will only explain the case \( \alpha > 0 \); the case \( \alpha = 0 \) is similar.

Choose bases of \( M_1 \) and \( M_2 \) and let \( A \) be the \( n \times n \) matrix defining \( \psi \); then \( A \) must be invertible over \( K\langle \gamma/t, t/\beta \rangle \). Choose \( \rho \in [\gamma, \beta] \); since \( \det(A) \) is a unit in \( K\langle \gamma/t, t/\beta \rangle \), we can find an invertible \( n \times n \) matrix \( W \) over \( K\langle \gamma/t, t/\beta \rangle \) such that \( \det(WA) = 1 \). (For instance, take \( W = \text{Diag}(\det(A)^{-1}, 1, \ldots, 1) \).)

It is then possible (see exercises) to find invertible matrices \( U, V \) over \( K[t, t^{-1}] \) such that \( |UWAV - I_n|_\rho < 1 \). By changing the initial choices of bases, we can force ourselves into the case \( |A - I_n|_\rho < 1 \).

By using the master slope factorization in the matrix ring over \( K\langle \gamma/t, t/\beta \rangle \), we can split \( A \) as a product of an invertible matrix over \( K\langle t/\beta \rangle \) and an invertible matrix over \( K\langle \gamma/t \rangle \). Using these to change basis in \( M_1 \) and \( M_2 \), respectively, we can put ourselves in the situation where \( A = I_n \), in which case we may identify the bases of \( M_1 \) and \( M_2 \). Take \( M \) to be the free module over \( K\langle \alpha/t, t/\delta \rangle \) with the same basis. \( \square \)

### 4 Notes

The Hadamard three circles theorem (Proposition 2(c)) is a special case of the fact that the **Shilov boundary** of the annulus \( \alpha \leq |t| \leq \beta \) consists of the two circles \(|t| = \alpha \) and \(|t| = \beta \). For much amplification of this remark, including a full-blown theory of harmonic functions.
on Berkovich analytic curves, see [Thu05]. For an alternate presentation, restricted to the Berkovich projective line but otherwise more detailed, see [BR07].

The patching lemma (Lemma 7) is a special case of the glueing property of coherent sheaves on affinoid rigid analytic spaces, i.e., the theorems of Kiehl and Tate [BGR84, Theorems 8.2.1/1 and 9.4.2/3]. The factorization argument in the proof, however, is older still; it is the nonarchimedean version of what is called a Birkhoff factorization over an archimedean field.

5 Exercises

1. Prove Proposition 3. (Hint: it may be easiest to first construct the limit using a single $\rho \in [\alpha, \beta]$, then show that it must also work for the other $\rho$.)

2. Let $R$ be the ring of formal power series over $K$ which converge for $|t| < 1$. Prove that $R$ is not noetherian; this is why I avoided introducing it.

3. Suppose $K$ is complete for a discrete valuation. Prove that any element of $\mathfrak{o}_K\llbracket t \rrbracket \otimes_{\mathfrak{o}_K} K$ (that is, a power series with bounded coefficients) is equal to a polynomial in $t$ times a unit. Then prove that this fails if $K$ is complete for a nondiscrete valuation.

4. Let $A$ be an $n \times n$ matrix over $K\langle \rho/t, t/\rho \rangle$ such that $|\det(A) - 1|_\rho < 1$. Prove that there exist invertible matrices $U, V$ over $K[t, t^{-1}]$ such that $|U^{-1}AV - I_n|_\rho < 1$. (Hint: perform approximate Gaussian elimination. An analogous argument, but in more complicated notation, is [Ked04, Lemma 6.2]. We will see a similar result in the unit on numerical analysis.)