CHAPTER 21

\textbf{\textit{p}}-adic Hodge theory

In this chapter, we describe an analogue of the construction of Chapter 17 for \textit{p}-adic representations of the absolute Galois group of a mixed characteristic local field. Beware that our presentation is historically inaccurate; see the notes.

**Hypothesis 21.0.1.** Throughout this chapter, let $K$ be a finite extension of $\mathbb{Q}_p$, let $V$ be a finite dimensional $\mathbb{Q}_p$-vector space, and let $\tau : G_K \to \text{GL}(V)$ be a continuous homomorphism for the \textit{p}-adic topology on $V$.

1. A few rings

**Definition 21.1.1.** Put $K_n = K(\zeta_{p^n})$ and $K_\infty = \bigcup_n K_n$. Let $F = \text{Frac} W(\kappa_K)$ and $F'$ be the maximal subfields of $K$ and $K_\infty$, respectively, which are unramified over $\mathbb{Q}_p$. Put $H_K = G_{K_\infty}$ and $\Gamma_K = G_{K_\infty}/K = G_K/H_K$.

**Definition 21.1.2.** Put $\mathfrak{o} = \mathfrak{o}_{C_p}$. Let $\breve{\mathfrak{E}}^+$ be the inverse limit of the system

\[ \cdots \to \mathfrak{o}/p\mathfrak{o} \to \mathfrak{o}/p\mathfrak{o} \]

in which each map is the \textit{p}-power Frobenius (which is a ring homomorphism). More explicitly, the elements of $\breve{\mathfrak{E}}^+$ are sequences $(x_0, x_1, \ldots)$ of elements of $\mathfrak{o}/p\mathfrak{o}$ for which $x_{n+1} = x_n$ for all $n$. In particular, for any nonzero $x \in \breve{\mathfrak{E}}^+$, the quantity $p^n v_p(x_n)$ is the same for all $n$ for which $x_n \neq 0$; we call this quantity $v(x)$, and put conventionally $v(0) = +\infty$. Choose $\epsilon = (\epsilon_0, \epsilon_1, \ldots) \in \breve{\mathfrak{E}}^+$ with $\epsilon_0 = 1$ and $\epsilon_1 \neq 1$.

The following observations were made by Fontaine and Wintenberger [FW79].

**Proposition 21.1.3.** The following are true.

(a) The ring $\breve{\mathfrak{E}}^+$ is a domain in which $p = 0$, with fraction field $\breve{\mathfrak{E}} = \breve{\mathfrak{E}}^+ [\epsilon^{-1}]$.

(b) The function $v : \breve{\mathfrak{E}}^+ \to [0, +\infty]$ extends to a valuation on $\breve{\mathfrak{E}}$, under which $\breve{\mathfrak{E}}$ is complete and $\mathfrak{o}_{\breve{\mathfrak{E}}} = \breve{\mathfrak{E}}^+$.

(c) The field $\breve{\mathfrak{E}}$ is the algebraic closure of $\kappa_K((\epsilon - 1))$. (The embedding of $\kappa_K((\epsilon - 1))$ into $\breve{\mathfrak{E}}$ exists because $v(\epsilon - 1) = p/(p - 1) > 0$.)

**Definition 21.1.4.** Let $\breve{\mathfrak{A}}$ be the ring of Witt vectors of $\breve{\mathfrak{E}}$, i.e., the unique complete discrete valuation ring with maximal ideal $p$ and residue field $\breve{\mathfrak{E}}$. The uniqueness follows from the fact that $\breve{\mathfrak{E}}$ is algebraically closed, hence perfect. In particular, the \textit{p}-power Frobenius on $\breve{\mathfrak{E}}$ lifts to an automorphism $\phi$ of $\breve{\mathfrak{A}}$.

**Definition 21.1.5.** Each element of $\breve{\mathfrak{A}}$ can be uniquely written as a sum $\sum_{n=0}^{\infty} p^n [x_n]$, where $x_n \in \breve{\mathfrak{E}}$ and $[x_n]$ denotes the Teichmüller lift of $x_n$ (the unique lift of $x_n$ that has a $p^m$-th root in $\breve{\mathfrak{A}}$ for all positive integers $m$); note that $\phi([x]) = [x^p] = [x]^p$. We may thus
equip $\tilde{A}$ with a weak topology, in which a sequence $x_m = \sum_{n=0}^{\infty} p^n [x_{m,n}]$ converges to zero if for each $n$, $v(x_{m,n}) \to \infty$ as $m \to \infty$. Let $A_{\mathbb{Q}_p}$ be the completion of $\mathbb{Z}_p[[(\epsilon) - 1]]$ in $\tilde{A}$ for the weak topology; as a topological ring, it is isomorphic to the ring $\mathfrak{o}_p$ defined over the base field $\mathbb{Q}_p$, with its own weak topology. It is also $\phi$-stable because $\phi((\epsilon)) = |\epsilon|^p$.

**Definition 21.1.6.** Let $A$ be the completion of the maximal unramified extension of $A_{\mathbb{Q}_p}$, viewed as a subring of $\tilde{A}$. Put

$$A_K = (A \cap \tilde{B})^{H_K},$$

where the right side makes sense because we have made all the rings so far in a functorial fashion, so that they indeed carry a $G_K$-action. Note that $A_K$ can be written as a ring of the form $\mathfrak{o}_E$, but with coefficients in $K'$ rather than in $\mathbb{Q}_p$.

**Definition 21.1.7.** For any ring denoted with a boldface $A$ so far, define the corresponding ring with $A$ replaced by $B$ by tensoring over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. For instance, $\tilde{B} = \tilde{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the fraction field of $\tilde{A}$.

## 2. $(\phi, \Gamma)$-modules

We are now ready to describe the mechanism, introduced by Fontaine, for converting Galois representations into modules over various rings equipped with much simpler group actions.

**Definition 21.2.1.** Recall that $V$ is a finite-dimensional vector space equipped with a continuous $G_K$-action. Put

$$D(V) = (V \otimes_{\mathbb{Q}_p} B)^{H_K};$$

by Hilbert’s Theorem 90, $D(V)$ is a finite dimensional $B_K$-vector space, and the natural map $D(V) \otimes_{B_K} B \to V \otimes_{\mathbb{Q}_p} B$ is an isomorphism. Since we only took $H_K$-invariants, $D(V)$ retains a semilinear action of $G_K/H_K = \Gamma_K$; it also inherits an action of $\phi$ from $B$. That is, $D(V)$ is a $(\phi, \Gamma)$-module over $B_K$, i.e., a finite free $B_K$-module equipped with semilinear $\phi$ and $\Gamma_K$-actions which commute with each other. It is also étale, which is to say the $\phi$-action is étale (unit-root); as in Definition 17.2.5, this is because one can find a $G_K$-invariant lattice in $V$.

**Theorem 21.2.2 (Fontaine).** The functor $D$, from the category of continuous representations of $G_K$ on finite dimensional $\mathbb{Q}_p$-vector spaces to the category of étale $(\phi, \Gamma)$-modules over $B_K$, is an equivalence of categories.

**Proof.** From $D(V)$, one can recover

$$V = (D(V) \otimes_{B_K} B)^{\phi=1}. \quad \square$$

Theorem 21.2.2 was refined by Cherbonnier and Colmez as follows [CC98].

**Definition 21.2.3.** Let $B_{\mathbb{Q}_p}^\dagger$ be the image of $E^\dagger$ under the identification of $E$ (with coefficients in $\mathbb{Q}_p$) with $B_{\mathbb{Q}_p}$ sending $t$ to $[\epsilon] - 1$. Let $B_K^\dagger$ be the integral closure of $B_{\mathbb{Q}_p}^\dagger$ in $B_K$. Again, $B_K^\dagger$ carries actions of $\phi$ and $\Gamma_K$. 172
**Definition 21.2.4.** Let \( A^\dagger \) be the set of \( x = \sum_{n=0}^{\infty} P^n [x_n] \in \tilde{A} \) such that \( \liminf_{n \to \infty} \{ v(x_n)/n \} > -\infty \). Define
\[
D^\dagger(V) = (V \otimes_{\Q_p} B^\dagger)^{HK};
\]
it is an étale \((\phi, \Gamma)\)-module over \( B^\dagger_K \).

The following is the main result of \([CC98]\).

**Theorem 21.2.5 (Cherbonnier-Colmez).** The functor \( D^\dagger \), from the category of continuous representations of \( G_K \) on finite dimensional \( \Q_p \)-vector spaces to the category of étale \((\phi, \Gamma)\)-modules over \( B^\dagger_K \), is an equivalence of categories.

**Remark 21.2.6.** By Theorem 21.2.2, it suffices to check that the base extension functor from étale \((\phi, \Gamma)\)-modules over \( B^\dagger_K \) to étale \((\phi, \Gamma)\)-modules over \( B_K \) is an equivalence. The full faithfulness of this functor is elementary; it follows from Lemma 18.4.6. The essential surjectivity is much deeper; it amounts to the fact that the natural map
\[
D^\dagger(V) \otimes_{B^\dagger_K} B^\dagger \rightarrow V \otimes_{\Q_p} B^\dagger
\]
is an isomorphism. Verifying this requires developing an appropriate analogy to Sen’s theory of decompletion; this analogy has been developed into a full abstract Sen theory by Berger and Colmez \([BC07]\).

A further variant was proposed by Berger \([Brg02]\).

**Definition 21.2.7.** Using the identification \( B^\dagger_{Q_p} \cong E^\dagger \), put
\[
B^\dagger_{\text{rig}, K} = B^\dagger_K \otimes_{B^\dagger_{Q_p}} \mathcal{R}.
\]
Note that \( B^\dagger_{\text{rig}, K} \) admits continuous extensions (for the LF-topology) of the actions of \( \phi \) and \( \Gamma_K \). Define
\[
D^\dagger_{\text{rig}}(V) = D^\dagger(V) \otimes_{B^\dagger_K} B^\dagger_{\text{rig}, K};
\]
it is an étale \((\phi, \Gamma)\)-module over \( B^\dagger_{\text{rig}, K} \).

**Theorem 21.2.8 (Berger).** The functor \( D^\dagger_{\text{rig}} \), from the category of continuous representations of \( G_K \) on finite dimensional \( \Q_p \)-vector spaces to the category of étale \((\phi, \Gamma)\)-modules over \( B^\dagger_{\text{rig}, K} \), is an equivalence of categories.

**Remark 21.2.9.** The principal content in Theorem 21.2.8 is that the base extension functor from étale \( \phi \)-modules over \( E^\dagger \) to étale \( \phi \)-modules over \( \mathcal{R} \) is fully faithful; this is elementary (see exercises). The essential surjectivity of the functor is almost trivial, since étaleness of the \( \phi \)-action is defined over the Robba ring by base extension from \( E^\dagger \). One needs only check that the \( \Gamma_K \)-action also descends to any étale lattice, but this is easy (it is similar to Lemma 18.4.4).

3. Galois cohomology

Since the functor \( D \) and its variants lose no information about Galois representations, it is unsurprising that they can be used to recover basic invariants of a representation, such as Galois cohomology.
**Definition 21.3.1.** Assume for simplicity that \( \Gamma_K \) is procyclic; this only eliminates the case where \( p = 2 \) and \( \{ \pm 1 \} \subset \Gamma \), for which see Remark 21.3.2 below. Let \( \gamma \) be a topological generator of \( \Gamma \). Define the Herr complex over \( B_K \) associated to \( V \) as the complex (with the first nonzero term placed in degree zero)

\[
0 \to D(V) \to D(V) \oplus D(V) \to D(V) \to 0
\]

with the first map being \( m \mapsto ((\phi - 1)m, (\gamma - 1)m) \) and the second map being \( (m_1, m_2) \mapsto (\gamma - 1)m_1 - (\phi - 1)m_2 \). (The fact that this is a complex follows from the commutativity between \( \phi \) and \( \gamma \).) Similarly, define the Herr complex over \( B_K^\dagger \) or \( B_{\text{rig}, K}^\dagger \) by replacing \( D(V) \) by \( D(V) \) or \( D_{\text{rig}}(V) \), respectively.

**Remark 21.3.2.** A more conceptual description, which also covers the case where \( \Gamma_K \) need not be profinite, is that one takes the total complex associated to

\[
0 \to C^\cdot(\Gamma_K, D(V))^{\phi - 1} \to C^\cdot(\Gamma_K, D(V)) \to 0.
\]

One might think of this as the “monoid cohomology” of \( \Gamma_K \times \phi^{\geq 0} \) acting on \( D(V) \).

**Theorem 21.3.3.** The cohomology of the Herr complex computes the Galois cohomology of \( V \).

**Proof.** For \( B_K \), the desired result was established by Herr [Her98]. The argument proceeds in two steps. One first takes cohomology of the Artin-Schreier sequence

\[
0 \to \mathbb{Q}_p \to B \xrightarrow{\phi - 1} B \to 0
\]

after tensoring with \( V \). This reduces the claim to the fact that the inflation homomorphisms

\[
H^i(\Gamma_K, D(V)) \to H^i(G_K, V \otimes_{\mathbb{Q}_p} B)
\]

are bijections; this is proved by adapting a technique introduced by Sen.

For \( B_K^\dagger \) and \( B_{\text{rig}, K}^\dagger \), the desired result was established by Liu [Liu07]; this proceeds by comparison with the original Herr complex rather than by imitating the above argument, though one could probably do that also.

**Remark 21.3.4.** As is done in [Her98, Liu07], one can make Theorem 21.3.3 more precise. For instance, the construction of Galois cohomology is functorial; there is also an interpretation in the Herr complex of the cup product in cohomology.

**Remark 21.3.5.** One can also use the Herr complex to recover Tate’s fundamental theorems about Galois cohomology (finite dimensionality, Euler-Poincaré characteristic formula, local duality). This was done by Herr in [Her01].

4. **Differential equations from \((\phi, \Gamma)\)-modules**

One of the original goals of \( p \)-adic Hodge theory was to associate finer invariants to \( p \)-adic Galois representations, so as for instance to distinguish those representations which arose in geometry (i.e., from the étale cohomology of varieties over \( \bar{K} \)). This was originally done using a collection of “period rings” introduced by Fontaine; more recently, Berger’s work has demonstrated that one can reproduce these constructions using \((\phi, \Gamma)\)-modules. Here is a brief description of an example that shows the relevance of \( p \)-adic differential equations to

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this study. We will make reference to Fontaine’s rings \(B_{dR}, B_{st}\) without definition, for which see [Brg04].

**Definition 21.4.1.** Let \(\chi : \Gamma_K \to \mathbb{Z}_p^\times\) denote the cyclotomic character; that is, for all nonnegative integers \(m\) and all \(\gamma \in \Gamma_K\),

\[
\gamma(\zeta_p^m) = \zeta_p^{\chi(\gamma)}.
\]

For \(\gamma \in \Gamma_K\) sufficiently close to 1, we may compute

\[
\nabla = \frac{\log(\gamma)}{\log \chi(\gamma)}
\]

as an endomorphism of \(D(V)\), using the power series for \(\log(1 + x)\). The result does not depend on \(\gamma\).

**Remark 21.4.2.** If one views \(\Gamma_K\) as a one-dimensional \(p\)-adic Lie group over \(\mathbb{Z}_p\), then \(\nabla\) is the action of the corresponding Lie algebra.

**Definition 21.4.3.** Note that \(\nabla\) acts on \(B^{\dagger \mathrm{rig}, K}\) with respect to

\[
f \mapsto [\epsilon] \log[\epsilon] \frac{df}{d[\epsilon]}.
\]

As a result, it does not induce a differential module structure with respect to \(\frac{d}{dt}\) on \(D(V)\), but only on \(D(V) \otimes B^{\dagger \mathrm{rig}, K}[\log[\epsilon]]^{-1}\). We say that \(V\) is **de Rham** if there exists a differential module with Frobenius structure \(M\) over \(B^{\dagger \mathrm{rig}, K}\) and an isomorphism

\[
D(V) \otimes B^{\dagger \mathrm{rig}, K}[\log[\epsilon]]^{-1} \to M \otimes B^{\dagger \mathrm{rig}, K}[\log[\epsilon]]^{-1}
\]

of differential modules with Frobenius structure.

One then has the following results of Berger [Brg02].

**Theorem 21.4.4 (Berger).** (a) The representation \(V\) is **de Rham** if and only if it is **de Rham** in Fontaine’s sense, i.e., if

\[
D_{dR}(V) = (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}
\]

satisfies

\[
D_{dR}(V) \otimes_K B_{dR} \cong V \otimes_{\mathbb{Q}_p} B_{dR}.
\]

(b) Suppose that \(V\) is **de Rham**. Then \(V\) is **semistable** in Fontaine’s sense, i.e.,

\[
D_{st}(V) = (V \otimes_{\mathbb{Q}_p} B_{st})^{G_K}
\]

satisfies

\[
D_{st}(V) \otimes_F B_{st} \cong V \otimes_{\mathbb{Q}_p} B_{st},
\]

if and only if there exists \(M\) as in Definition 21.4.3 which is unipotent.

Applying Theorem 18.1.8 then yields the following corollary, which was previously a conjecture of Fontaine [Fon94, 6.2].

**Corollary 21.4.5 (Berger).** Every **de Rham** representation is **potentially semistable**, i.e., becomes semistable upon restriction to \(G_L\) for some finite extension \(L\) of \(K\).
Remark 21.4.6. The term “de Rham” is meant to convey the fact that if $V = H^i_{\text{dR}}(X \times_K K^{\text{alg}}, \mathbb{Q}_p)$ for $X$ a smooth proper variety over $K$, then $V$ is de Rham and you can use the aforementioned constructions to recover $H^i_{\text{dR}}(X, K)$ functorially from $V$ (solving Grothendieck’s “problem of the mysterious functor”). See [Brg04] for more of the story.

5. Beyond Galois representations

The category of arbitrary $(\phi, \Gamma)$-modules over $B_{\text{rig}, K}^\dagger$ turns out to have its own representation-theoretic interpretation; it is equivalent to the category of $B$-pairs introduced by Berger [Brg07a]. One can associate “Galois cohomology” to such objects using the Herr complex; it has been shown by Liu [Liu07] that the analogues of Tate’s theorems (see Remark 21.3.5) still hold. These functors can be interpreted as the derived functors of $\text{Hom}(D_{\text{rig}}^\dagger(V_0), \cdot)$ for $V_0$ the trivial representation [Ked07f, Appendix].

One may wonder why one should be interested in $(\phi, \Gamma)$-modules over $B_{\text{rig}, K}^\dagger$ if ultimately one has in mind an application concerning only Galois representations. One answer is that converting Galois representations into $(\phi, \Gamma)$-modules exposes extra structure that is not visible without the conversion.

Definition 21.5.1 (Colmez). We say $V$ is trianguline if $D_{\text{rig}}^\dagger(V)$ is a successive extension of $(\phi, \Gamma)$-modules of rank 1 over $B_{\text{rig}, K}^\dagger$. The point is that these need not be étale, so $V$ need not be a successive extension of representations of dimension 1.

The trianguline representations have the dual benefits of being relatively easy to classify, and somewhat commonplace. On one hand, Colmez [Col07] classified the two-dimensional trianguline representations of $G_{\mathbb{Q}_p}$; the classification includes a parameter (the $L$-invariant) relevant to $p$-adic $L$-functions. On the other hand, a result of Kisin [Kis03] shows that the Galois representations associated to many classical modular forms are trianguline.

Notes

Our presentation here is largely a summary of Berger’s [Brg04], which we highly recommend.

A variant of the theory of $(\phi, \Gamma)$-modules was introduced by Kisin [Kis06], using the Kummer tower $K(p^{1/p^n})$ instead of the cyclotomic tower $K(\zeta_{p^n})$. This leads to certain advantages, particularly when studying crystalline representations. Kisin’s work is based on an earlier paper of Berger [Brg07b]; both of these use slope filtrations (as in Theorem 18.4.1) to recover a theorem of Colmez-Fontaine classifying semistable Galois representations in terms of certain linear algebraic data.

After [Brg02] appeared, Fontaine succeeded in giving a direct proof of Corollary 21.4.5 (i.e., not going through $p$-adic differential equations). We do not have a reference for this.

Exercises

(1) (Compare [Tsu96, Proposition 2.2.2].) Let $A$ be an $n \times n$ matrix over $\mathfrak{o}_{\mathbb{F}_1}$, and suppose $v \in \mathbb{E}^n$, $w \in (\mathbb{E}^\dagger)^n$ satisfy $Av - \phi(v) = w$. Then $v \in (\mathbb{E}^\dagger)^n$. This gives a direct proof of some cases of Theorem 18.5.1, in the spirit of Lemma 18.4.6. (Hint: reduce to the case where $|A|_\rho \leq 1$ for some $\rho \in (0, 1)$ for which $|w|_\rho < \infty$. Then use $|w|_\rho$ to bound the terms of $v = \sum_i v_i t_i^\rho$ for which $|v_i| \geq c$.)