CHAPTER 20

Rigid cohomology

In this chapter, we introduce a bit of the theory of rigid p-adic cohomology, as developed by Berthelot and others. In particular, we illustrate the role played by the p-adic local monodromy theorem in a fundamental finiteness problem in the theory.

1. Isocrystals on the affine line

In this section, we recall Crew's interpretation [**Cre98**] of overconvergent F-isocrystals on the affine line and their cohomology.

Definition 20.1.1. Let k be a perfect (for simplicity) field of characteristic $p > 0$. Let K be a complete discrete (again for simplicity) nonarchimedean field of characteristic zero with $\kappa_K = k$. An *overconvergent F-isocrystal* on the affine line over k (with coefficients in K) is a finite differential module with Frobenius structure on the ring $\mathcal{A} = \cup_{\beta>1} K\langle t/\beta\rangle$, for some absolute Frobenius lift ϕ; as in Proposition 15.3.1, the resulting category is independent of the choice of the Frobenius lift.

Definition 20.1.2. Let M be an overconvergent F-isocrystal on the affine line over k. Let $\mathcal{R}$ be a copy of the Robba ring with series parameter t^{-1}, so that we can identify $\mathcal{A}$ as a subring of $\mathcal{R}$. Define

$$\begin{aligned} H^0(\mathbb{A}^1_k, M) &= \ker(D, M) \\ H^1(\mathbb{A}^1_k, M) &= \operatorname{coker}(D, M) \\ H^0_{\mathrm{loc}}(\mathbb{A}^1_k, M) &= \ker(D, M \otimes_{\mathcal{A}} \mathcal{R}) \\ H^1_{\mathrm{loc}}(\mathbb{A}^1_k, M) &= \operatorname{coker}(D, M \otimes_{\mathcal{A}} \mathcal{R}) \\ H^1_c(\mathbb{A}^1_K, M) &= \ker(D, M \otimes_{\mathcal{A}} (\mathcal{R}/\mathcal{A})) \\ H^2_c(\mathbb{A}^1_K, M) &= \operatorname{coker}(D, M \otimes_{\mathcal{A}} (\mathcal{R}/\mathcal{A})). \end{aligned}$$

By taking kernels and cokernels in the short exact sequence

$$0 \to M \to M \otimes_{\mathcal{A}} \mathcal{R} \to M \otimes_{\mathcal{A}} (\mathcal{R}/\mathcal{A}) \to 0$$

and applying the snake lemma, we get an exact sequence

$$0 \to H^0(\mathbb{A}^1_k, M) \to H^0_{\mathrm{loc}}(\mathbb{A}^1_k, M) \to H^1_c(\mathbb{A}^1_k, M) \to H^1(\mathbb{A}^1_k, M) \to H^1_{\mathrm{loc}}(\mathbb{A}^1_k, M) \to H^2_c(\mathbb{A}^1_k, M) \to 0.$$

Remark 20.1.3. Crew shows [**Cre98**] that in this construction, H^i computes the *rigid cohomology* of M, H^i_c computes the *rigid cohomology with compact supports*, and H^i_{loc} computes some sort of local cohomology at ∞.

Crew's main result in this setting is the following.

Theorem 20.1.4 (Crew). *The spaces $H^i(\mathbb{A}^1_k, M)$, $H^i_c(\mathbb{A}^1_k, M)$, $H^i_{\mathrm{loc}}(\mathbb{A}^1_K, M)$ are all finite dimensional over K. Moreover, the Poincaré pairings*

$$H^i(\mathbb{A}^1_k, M) \times H^{2-i}_c(\mathbb{A}^1_k, M^\vee) \to H^2_c(\mathbb{A}^1_k, \mathcal{A}) \cong K$$
$$H^i_{\mathrm{loc}}(\mathbb{A}^1_k, M) \times H^{1-i}_{\mathrm{loc}}(\mathbb{A}^1_k, M^\vee) \to H^1_{\mathrm{loc}}(\mathbb{A}^1_K, \mathcal{A}) \cong K$$

are perfect.

The key ingredient is the fact that $M \otimes \mathcal{R}$ is quasiunipotent by the p-adic local monodromy theorem (Theorem 18.1.8), which implies finiteness of $H^i_{\mathrm{loc}}(\mathbb{A}^1_k, M)$. This implies the finite dimensionalities except for $H^1_c(\mathbb{A}^1_k, M)$ and $H^1(\mathbb{A}^1_k, M)$; however, these are related by a map with finite dimensional kernel and cokernel. Moreover, they carry incompatible topologies: the former is a Fréchet space, while the latter is dual to a Fréchet space. This incompatibility can only be resolved by both spaces being finite dimensional.

2. Consequences in rigid cohomology

The previous construction extends, with some work, to a theory of rigid cohomology with/without compact supports on arbitrary varieties over k, with coefficients in overconvergent F-isocrystals. For constant coefficients, it was shown by Berthelot [**Brt97a, Brt97b**] that this theory has all of the desired properties of a Weil cohomology: finite dimensionality, Poincaré duality, Künneth formula, cycle class maps, etc. Using a relative version of Theorem 20.1.4, one can extend these to nonconstant coefficients [**Ked06a**].

The analogy with étale cohomology with ℓ-adic coefficients is tempting, and indeed motivates most of the preceding development, but remains somewhat imperfect. Most notably, overconvergent F-isocrystals in rigid cohomology are analogous only to lisse (smooth) ℓ-adic sheaves, whereas for most serious computations one needs also constructible sheaves (or some appropriate derived category thereof). There is a proposed theory of arithmetic $\mathcal{D}$-modules that would play the appropriate p-adic role, but this theory remains underdeveloped; see [**Brt02**].

Nonetheless, in the interim, one can still carry many good properties of ℓ-adic cohomology to the p-adic setting, e.g., Laumon's Fourier-theoretic reinterpretation of Deligne's second proof of the Weil conjectures [**Ked06b**]. It is hoped that one can go further, establishing some properties in p-adic cohomology that are only conjectural in ℓ-adic cohomology, such as Deligne's weight-monodromy conjecture.

3. Machine computations

In recent years, interest has emerged in explicitly computing the zeta functions of algebraic varieties defined over finite fields. Some of this interest has come from cryptography, particularly the use of Jacobians of elliptic (and later hyperelliptic) curves over finite fields as "black box abelian groups" for certain public-key cryptography schemes (Diffie-Hellman, ElGamal).

For elliptic curves, a good method for doing this was proposed by Schoof [**Sch85**]. It amounts to computing the trace of Frobenius on the ℓ-torsion points, otherwise known as the étale cohomology with $\mathbb{F}_\ell$-coefficients, for enough small values of ℓ to determine uniquely the one unknown coefficient of the zeta function within the range prescribed by the Hasse-Weil bound.

It turns out to be somewhat more difficult to execute Schoof's scheme for curves of higher genus, as discovered by Pila [**Pil90**]. One is forced to work with higher division polynomials in order to compute torsion of the Jacobian of the curve; the interpretation in terms of étale cohomology is of little value because the definition of étale cohomology is not intrinsically computable. (It is easy to write down cohomology classes, but it is difficult to test two such classes for equality.)

It was noticed by several authors that rigid cohomology is intrinsically more computable, and so lends itself better to this sort of task. Specifically, Kedlaya [**Ked03**] proposed an algorithm using rigid cohomology (in its guise for smooth affine varieties, known as Monsky-Washnitzer cohomology) for computing the zeta function of a hyperelliptic curve over a finite field of small odd characteristic. The limitation to odd characteristic was lifted by Denef and Vercauteren [**DV06**]; the limitation to small characteristic was somewhat remedied by Harvey [**Har07**], who improved the dependence on the characteristic p from $O(p)$ to $O(p^{1/2+\epsilon})$.

More recently, interest has emerged in considering also higher-dimensional varieties, partly come from potential applications in the study of mirror symmetry for Calabi-Yau varieties. In this case, étale cohomology is of no help at all, since there is no geometric interpretation of H^i_{et} for $i > 1$ analogous to the interpretation for $i = 1$ in terms of the Jacobian. Rigid cohomology should still be computable, but relatively little progress has been made in making these computations practical (one exception being the treatment of smooth surfaces in projective 3-space in [**AKR07**]). It may be necessary to combine these techniques with Lauder's deformation method (see Remark 19.2.2) for best results.

Notes

Until recently, while there were some useful survey articles about rigid cohomology (e.g., Berthelot's [**Brt86**]), and some fragmentary foundational materials (e.g., Berthelot's [**Brt96**]), there was no comprehensive introductory text on the subject. That state of affairs has been remedied by the appearance of the book of le Stum [**leS07**]; this book may be particularly helpful for those interested in machine calculations.

Crew's work, and subsequent work which builds on it (e.g., [**Ked06a**]), makes essential use of nonarchimedean functional analysis, as is evident in the discussion of Theorem 20.1.4. We recommend Schneider's book [**Sch02**] as a friendly introduction to this topic.

As a companion to our original paper on hyperelliptic curves [**Ked03**], we recommend Edixhoven's course notes [**Edi06**]; some discussion is also included in [**FvdP04**, Chapter 7]. We gave a high-level summary of the general approach in [**Ked04c**].