Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lectures 6-8: Lattices September 10, 12 and 14, 2001

Reminder. If you didn't submit a questionnaire, please at least give me your email address (preferably by sending me an email at `kedlaya@math`). I'll use email and the course web site for some announcements.

Reference. Neukirch, Sections 1.4 and 1.5.

Jargon watch. Warning: when I say "lattice", I usually mean what the book calls a "complete lattice". If not, I'll say "partial lattice". Also, I will use the more common term *fundamental domain* for what the book calls a *fundamental mesh*, that is, a subset of a real vector space whose translates by some lattice exactly cover the vector space.

A *real embedding* of a number field K is an injection $K \hookrightarrow \mathbb{R}$. A *complex embedding* is an injection $K \hookrightarrow \mathbb{C}$ whose image is not contained in $\mathbb{R}$. Complex embeddings come in pairs that are interchanged by applying complex conjugation to $\mathbb{C}$; the *signature* of K is the pair (r, s), where r is the number of real embeddings and s is the number of pairs of complex embeddings.

Outline of lectures.

1. Define a (complete) lattice in several different ways:
   (a) as a subgroup of a real vector space generated by a basis;
   (b) as a discrete subgroup of a real vector space;
   (c) as a free abelian group with a positive definite inner product;
   (d) as a free abelian group with a positive norm.
2. Define the volume of a lattice. Note that passing to a sublattice of index n multiplies the volume by n.
3. State Minkowski's lattice theorem: any centrally symmetric, convex subset of $\mathbb{R}^n$ of volume greater than 2^n contains a nonzero element of any lattice of volume 1.
4. Deduce from Minkowski's theorem that a rank n lattice of volume V contains a nonzero vector of norm at most $2(V/B_n)^{1/n}$, where B_n is the volume of the n-ball.
5. Define real embeddings, complex embeddings, and the signature of a number field.
6. Given a number field, define the Minkowski space (as the product of the real and complex embeddings).
7. Reinterpret the trace pairing as the Minkowski metric, and the discriminant of a basis as its Minkowski volume.
8. Define the multiplicative Minkowski space (to be used later).