Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lectures 9-10: The Class Number
September 17 and 19, 2001

Reference. Neukirch, Sections 1.5 and 1.6.

Jargon watch. The *absolute norm* norm of an integral ideal $\mathfrak{a}$ is equal to the index $[\mathfrak{o}_K : \mathfrak{a}]$, i.e., the number of elements of the quotient ring $\mathfrak{o}_K/\mathfrak{a}$. (Warning: the book uses parentheses instead of brackets to denote the index.) Note that the absolute norm of a principal ideal (α) is equal to the absolute value of $\mathrm{Norm}(\alpha)$.

Outline of lectures.

1. Define the absolute norm of an integral ideal.
2. Check that the absolute norm is multiplicative, and that for a principal ideal it equals the absolute value of the norm of a generator. Define absolute norm for fractional ideals too.
3. Note that the number of ideals of bounded absolute norm is finite.
4. Prove Theorem 5.3: every ideal contains an element which is "small" in each embedding (relative to the absolute norm of the ideal). (This uses Minkowski's lattice point theorem, applied to the Minkowski space of the number field.)
5. Conclude that every ideal contains an element of "small" norm (Lemma 6.2).
6. Show that every ideal class contains an ideal of "small" absolute norm.
7. Conclude that the number of ideal classes in a number field K is finite. Define the *class number* of K to be the number of ideal classes.
8. Make some comments about class numbers, e.g., of quadratic fields.