Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lectures 11-12: Dirichlet's Units Theorem
September 21 and 24, 2001

Reference. Neukirch, Sections 1.5 and 1.7.

Outline of lectures.

1. State Dirichlet's Units Theorem: the group of units of $\mathfrak{o}_K$ is isomorphic to the finite group of roots of unity times a free abelian group of rank $r+s-1$, where (r,s) is the signature of K.

2. Examples: an imaginary quadratic field has finitely many units. A real quadratic field has a rank 1 group of units; point out relationship to Pell's equation (see also homework).

3. Define the logarithm map $K^* \to \prod_\rho \mathbb{R} \times \prod_\sigma \mathbb{R}$, which on the factor corresponding to an embedding τ carries α to $\log|\tau(\alpha)|$. If K has signature (r,s), the image is a real vector space of dimension $r+s-1$.

4. Observe that the kernel of the logarithm map on $\mathfrak{o}_K$ consists of algebraic integers all of whose conjugates in $\mathbb{C}$ lie on the unit circle. We have seen that there are finitely many of these in K, so they are roots of unity.

5. Note that the units of $\mathfrak{o}_K$ (modulo the roots of unity in K) map into the subspace H of the image of the logarithm map consisting of points with sum of coordinates 0. More precisely, the sum of coordinates of the image of α is $\log|\operatorname{Norm}(\alpha)|$.

6. Show that the image of the units of $\mathfrak{o}_K$ is a discrete subgroup of H, i.e., a subgroup generated by at most $r+s-1$ independent vectors in H.

7. Show that the image of the units of $\mathfrak{o}_K$ is a *complete* lattice in H, by showing that every element of H can be translated by a unit into a bounded region. In fact, we'll do this not in H itself, but its preimage in Minkowski space, the "norm one surface", using Minkowski's lattice point theorem on the integers in Minkowski space.

8. Deduce the Units Theorem, using the classification of finitely generated abelian groups.

9. Note: the space H inherits a norm from the usual norm on $\mathbb{R}^{r+s}$, so we can define the volume of the lattice of units, called the *regulator*. We will not need this for the moment (but it's closely related to the zeta function of a number field).