**Math 254A, UC Berkeley, Fall 2001 (Kedlaya)**
**Lectures 15-16: Completions and Local Fields**
**October 1 and 3, 2001**

**Reference**. Neukirch, Sections 2.4 and 2.5, and a bit of 2.6.

**Jargon watch**. An exponential valuation is *discrete* if it takes values in a discrete subgroup of $\mathbb{R}$, which can always be normalized to be $\mathbb{Z}$ itself. A valuation is *discrete* if it is nonarchimedean and comes from a discrete exponential valuation. (For example, the $p$-adic valuation on $\mathbb{Q}$ is discrete.)

Recall that given a field with a nonarchimedean valuation, the *valuation ring* of the field is the set of elements of norm at most 1.

**Outline of lectures**.

1. For $K$ a number field and $\mathfrak{p}$ a nonzero prime ideal, define $K_\mathfrak{p}$ as the completion of $K$ with respect to the valuation $|\cdot|_\mathfrak{p}$. If $\mathfrak{p} \cap \mathbb{Q} = (p)$, then $[K_\mathfrak{p} : \mathbb{Q}_p] \leq [K : \mathbb{Q}]$, but the inequality can be strict.

2. Demonstrate some examples for the Gaussian numbers $\mathbb{Q}(i)$, with $\mathfrak{p} = (2+i)$, $\mathfrak{p} = (3)$, and the "anomalous" case $\mathfrak{p} = (1+i)$.

3. Define a *local field* as a field which is complete with respect to a discrete valuation, with finite residue field. Note: the *only* such fields are finite extensions of $\mathbb{Q}_p$ or of the power series field $\mathbb{F}_p((t))$.

4. Focus a bit on discrete valuation rings. (Remember, these aren't necessarily complete.) Note that the maximal ideal $\mathfrak{m}$ of a DVR is generated by any element of $\mathfrak{m} \setminus \mathfrak{m}^2$, and every ideal of a DVR is a power of the maximal ideal. In particular, a DVR is a principal ideal domain. For $U_i$ the group of units congruent to 1 modulo $\pi^i$, point out the isomorphisms $U_0/U_1 \cong (R/\mathfrak{m})^*$ and $U_i/U_{i+1} \cong R/\mathfrak{m}$ for $i \geq 1$.

5. State and prove Hensel's lemma: if a monic polynomial over the ring of integers in a local field factors, modulo the maximal ideal, into two coprime factors, then there is a corresponding factorization of the original polynomial. Example from last time: if $n$ is a quadratic residue modulo $p$, then $x^2 - n$ factors over $\mathbb{Z}_p$.

6. State the extension theorem: if $L/K$ is a finite extension of fields, and $K$ is complete with respect to the valuation $|\cdot|$, then there exists a unique extension of $|\cdot|$ to $L$. Sketch the proof in the nonarchimedean case (the archimedean case uses Ostrowski's theorem, see Neukirch 2.4).

7. Compare the previous theorem to the case $\mathbb{Q}(i)/\mathbb{Q}$ of an extension of number fields: a valuation downstairs may extend upstairs in multiple ways. Sneak preview: if $K$ is a number field and $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}_K$, the extensions of $|\cdot|_\mathfrak{p}$ to a finite extension $L$ of $K$ correspond to the prime factors of $\mathfrak{p}\mathfrak{o}_L$.

8. Define the Newton polygon of a polynomial over a discrete valuation ring. Show that the slopes of the Newton polygon coincide with the valuations of the roots of the polynomial.

9. If $R$ is the valuation ring of a local field, observe that any irreducible polynomial over $R$ only has a single slope in its Newton polygon (by the extension theorem).