Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lectures 18-19: Ramification
October 8 and 10, 2001

Reference. Neukirch, Sections 2.6 and 2.7.

Jargon watch. Reminder: a *local field* is a field which is complete with respect to a discrete nonarchimedean valuation. You may substitute "finite extension of $\mathbb{Q}_p$" if you like.

Outline of lectures.

1. (Left over from last time) Define the Newton polygon of a polynomial over a local field, and show that it computes the valuations of the roots of the polynomial. Note that an irreducible polynomial has one slope (by the extension theorem, all roots of the polynomial must have the same valuation).

2. Given a finite extension L/K of local fields, define the *ramification index* $e = e(L/K)$ as the index of the value group of K in the value group of L. That is, if π generates the maximal ideal of the valuation ring of L, then π^e generates the maximal ideal of the valuation ring of K.

3. Given a finite extension L/K of local fields, define the *inertia degree* $f = f(L/K)$ as the degree of the residue field of L over the residue field of K.

4. Prove that if K is a local field, then $e(L/K)f(L/K) = [L : K]$ (the "fundamental identity"). Illustrate this with $\mathbb{Q}_p(i)$ and possibly other examples.

5. Define an *unramified* extension L/K of local fields as one in which $e(L/K) = 1$; that is, the residue field extension has degree $[L : K]$. Note that if K'/K is any finite extension, then the compositum $K'L$ is unramified over K'. Also note that a local field has only one unramified extension of any given degree (see homework). At the other extreme, define a *totally ramified* extension to be one where $f(L/K) = 1$.

6. Distinguish between tamely ramified and wildly ramified extensions. An extension L/K of local fields, whose residue fields have characteristic p, is *tamely ramified* if its ramification degree is not divisible by p. Show that tamely ramified extensions are generated by radicals. An example of a wildly ramified extension: $\mathbb{Q}_p(p^{1/p})$.

7. Illustrate with the example $\mathbb{Q}_p(\zeta_n)$, where ζ_n is an n-th root of unity.