Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lecture 20: Ramification in Number Fields
October 12, 2001

Reference. Neukirch, Section 2.8, and to a lesser extent Section 1.8.

Outline of lectures.

1. (Left over from last time) Work out the structure of $\mathbb{Q}_p(\zeta_{p^n})$. In particular, show that $1 - \zeta_{p^n}$ generates the valuation ring.

2. Let L/K be an extension of *number fields*, let $\mathfrak{q}$ be a nonzero prime ideal of $\mathfrak{o}_L$, and let $\mathfrak{p} = K \cap \mathfrak{q}$. As noted earlier, the completion $L_\mathfrak{q}$ is a finite extension of $K_\mathfrak{p}$. Define the ramification index and inertia degree $e(\mathfrak{q}/\mathfrak{p}) = e(L_\mathfrak{q}/K_\mathfrak{p})$ and $f(\mathfrak{q}/\mathfrak{p}) = f(L_\mathfrak{q}/K_\mathfrak{p})$.

3. Prove that the $\mathfrak{p}$-adic valuation does not necessarily extend uniquely to L, in contrast to the case where K was a local field. Rather, the extensions are precisely the $\mathfrak{q}$-adic valuations for all primes $\mathfrak{q}$ containing $\mathfrak{o}_L\mathfrak{p}$, or equivalently, all primes of $\mathfrak{o}_L$ in the prime factorization of $\mathfrak{o}_L\mathfrak{p}$. (The point: given an extension of the valuation to L, the maximal ideal of the valuation ring is a prime ideal containing $\mathfrak{o}_L\mathfrak{p}$.)

4. For L/K a finite extension of number fields and $\mathfrak{p}$ a prime of $\mathfrak{o}_K$, prove the fundamental identity
$$[L : K] = \sum_\mathfrak{q} [L_\mathfrak{q} : K_\mathfrak{p}] = \sum_\mathfrak{q} e(\mathfrak{q}/\mathfrak{p}) f(\mathfrak{q}/\mathfrak{p}),$$
where the sum runs over all primes $\mathfrak{q}$ of $\mathfrak{o}_L$ containing $\mathfrak{p}$.

5. Point out another interpretation: the tensor product $L \otimes_K K_\mathfrak{p}$ factors as the product of the $L_\mathfrak{q}$.

6. Prove that $\mathfrak{o}_L\mathfrak{p} = \prod_\mathfrak{q} \mathfrak{q}^{e(\mathfrak{q}/\mathfrak{p})}$. That is, the ramification index is the factor to which $\mathfrak{q}$ shows up in the prime factorization of $\mathfrak{o}_L\mathfrak{p}$.

7. Discuss some examples, e.g., $\mathbb{Q}(i)$ and/or $\mathbb{Q}(x)/(x^3 + x + 1)$. Point out (to be proved later) that when $K = \mathbb{Q}$, the only primes of K over which ramification happens are the ones dividing the discriminant of L. (One part of this claim is on this week's homework.) Later we will define the "relative discriminant", which will allow to make a similar statement for general K.