Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lectures 21–22: Ramification and Splitting in Number Fields
October 22 and 24, 2001

Reference. Neukirch, Section 2.8, and to a lesser extent Section 1.8.

Notation. For L/K an extension of number fields, $\mathfrak{p}$ a prime of $\mathfrak{o}_K$ and $\mathfrak{q}$ a prime of $\mathfrak{o}_L$ containing $\mathfrak{p}\mathfrak{o}_L$, we defined last time the ramification index and inertia degree $e(\mathfrak{q}/\mathfrak{p}) = e(L_\mathfrak{q}/K_\mathfrak{p})$ and $f(\mathfrak{q}/\mathfrak{p}) = f(L_\mathfrak{q}/K_\mathfrak{p})$.

Outline of lectures.

1. For L/K a finite extension of number fields and $\mathfrak{p}$ a prime of $\mathfrak{o}_K$, prove the fundamental identity
$$[L:K] = \sum_\mathfrak{q} [L_\mathfrak{q} : K_\mathfrak{p}] = \sum_\mathfrak{q} e(\mathfrak{q}/\mathfrak{p}) f(\mathfrak{q}/\mathfrak{p}),$$
where the sum runs over all primes $\mathfrak{q}$ of $\mathfrak{o}_L$ containing $\mathfrak{p}$. Proof outline: write $L = K(\alpha)$ by the primitive element theorem, where α has minimal polynomial $P(x)$, and look at the factorization of $P(x)$ over $K_\mathfrak{p}$. Each factor corresponds to a prime $\mathfrak{q}$ of $\mathfrak{o}_L$ containing $\mathfrak{p}$.

2. Point out another interpretation: the tensor product $L \otimes_K K_\mathfrak{p}$ factors as the product of the $L_\mathfrak{q}$.

3. Prove that $\mathfrak{o}_L \mathfrak{p} = \prod_\mathfrak{q} \mathfrak{q}^{e(\mathfrak{q}/\mathfrak{p})}$. That is, the ramification index is the factor to which $\mathfrak{q}$ shows up in the prime factorization of $\mathfrak{o}_L\mathfrak{p}$.

4. Discuss some examples, e.g., $\mathbb{Q}(i)$ and/or $\mathbb{Q}(x)/(x^3+x+1)$.

5. As an additional example, mention the law of quadratic reciprocity (Theorem 8.6). To be done a bit later, maybe Friday: discuss the proof given in Neukirch section I.8 (using Gauss sums).

6. If $L = K(\alpha)$ for some $\alpha \in \mathfrak{o}_L$, define the *conductor* of α as the ideal of $\beta \in \mathfrak{o}_K$ such that $\beta\mathfrak{o}_K \subseteq \mathfrak{o}_K[\alpha]$. Note that if $\mathfrak{p}$ is coprime to the conductor of α, the ramification and inertia degrees are computed by the prime factorization of the minimal polynomial of α modulo $\mathfrak{p}$. (Namely, each factor of degree d corresponds to a prime above $\mathfrak{p}$ with inertia degree d, and its multiplicity is the ramification degree of the prime.)

7. Important special case: if $\mathfrak{o}_L = \mathfrak{o}_K[\alpha]$, then the conductor is the unit ideal in $\mathfrak{o}_K$, so the above criterion applies for all primes. This is useful for determining splitting, but pretty useless for determining ramification, since you need to compute discriminants to find the ring of integers in $\mathfrak{o}_L$; instead, you should factor the minimal polynomial of α over the completion $K_\mathfrak{p}$.

8. A more useful form of the splitting criterion: if L/K is an extension of number fields, $\alpha \in \mathfrak{o}_L$ generates L over K and $\mathfrak{p}$ is coprime to the discriminant of the minimal polynomial of α over K, then the degrees of the factors of the minimal polynomial modulo $\mathfrak{p}$ are the inertia degrees of the primes above $\mathfrak{p}$ in L, all of which are unramified.

9. If L/K is an extension of local fields, define the *relative discriminant* $\mathrm{Disc}(L/K)$ as the ideal of $\mathfrak{o}_K$ generated by the determinant of the matrix $\mathrm{Trace}(x_i x_j)_{i,j=1}^n$, for $x_1, \ldots, x_n$ a basis of $\mathfrak{o}_L$ over $\mathfrak{o}_K$. Caution: the relative discriminant is only defined as an ideal, not as an element.

10. If L/K is an extension of number field, define the relative discriminant

$$\mathrm{Disc}(L/K) = \prod_{\mathfrak{p}} \prod_{\mathfrak{q}} \mathrm{Disc}(L_{\mathfrak{q}}/K_{\mathfrak{p}}).$$

Here $\mathfrak{p}$ runs over all nonzero primes of $\mathfrak{o}_K$, $\mathfrak{q}$ runs over all primes of $\mathfrak{o}_L$ containing $\mathfrak{p}\mathfrak{o}_L$, and we interpret $\mathrm{Disc}(L_{\mathfrak{q}}/K_{\mathfrak{p}})$ to be a power of $\mathfrak{p}$ even though it's really a power of the maximal ideal of $K_{\mathfrak{p}}$.

11. Note that for $K = \mathbb{Q}$, the relative discriminant $\mathrm{Disc}(L/K)$ is the ideal generated by the usual discriminant of L.