Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lecture 24: Ramification in Galois Extensions
October 29, 2001

Reference. Neukirch, Section 1.9; optionally, Section 2.9 as well.

Jargon watch. Given an extension L/K of fields, the *normal closure* of L/K is the compositum of the image of L' under all elements of $\mathrm{Gal}(\overline{K}/K)$. If this equals L, we say L/K is *normal*. We say L/K is *Galois* if it is finite, normal and separable, or equivalently, if $\#\,\mathrm{Gal}(L/K) = [L : K]$. Note that finite and separable are automatic for L/K an extension either of number fields or of finite extensions of $\mathbb{Q}_p$.

Outline of lecture.

1. Let L/K be a Galois extension of number fields with Galois group G. Show that G acts transitively on the primes $\mathfrak{b}$ of $\mathfrak{o}_L$ lying above a prime $\mathfrak{p}$ of $\mathfrak{o}_K$. Note in particular that they all have the *same* ramification index and inertia degree over $\mathfrak{p}$.

2. Let $\mathfrak{b}$ be a prime of $\mathfrak{o}_L$ lying over the prime $\mathfrak{p}$ of $\mathfrak{o}_K$. Define the *decomposition group* $G_{\mathfrak{b}}$ of $\mathfrak{b}$ over K as the set of $\sigma \in G$ mapping $\mathfrak{b}$ to itself (but not necessarily by the identity map!). Define the fixed field of $G_{\mathfrak{b}}$ as the *decomposition field* of $\mathfrak{b}$ over K.

3. Interpret some of the splitting behavior of $\mathfrak{p}$ in terms of the decomposition group. E.g., $G_{\mathfrak{b}} = 1$ if and only if $\mathfrak{p}$ is totally split.

4. Show that $G_{\mathfrak{b}}$ surjects onto the Galois group $\mathrm{Gal}((\mathfrak{o}_L/\mathfrak{b})/(\mathfrak{o}_K/\mathfrak{p}))$. The kernel of this map is the *inertia group* $I_{\mathfrak{b}}$ of $\mathfrak{b}$ over K, whose fixed field is the *inertia field* of $\mathfrak{b}$ over K. Again, interpret splitting behavior in terms of this: the order of $I_{\mathfrak{b}}$ is the ramification index $e(\mathfrak{b}/\mathfrak{p})$, and the order of $G_{\mathfrak{b}}/I_{\mathfrak{b}}$ is the inertia degree.

5. Note that $G_{\mathfrak{b}}$ can also be interpreted as the Galois group of the extension $L_{\mathfrak{b}}/K_{\mathfrak{p}}$ of local fields, and that its subgroup $I_{\mathfrak{b}}$ is the Galois group of $L_{\mathfrak{b}}$ over its maximal unramified subextension (over $K_{\mathfrak{p}}$). (Of course, one can also define the inertia subgroup for any Galois extension of local fields.)

6. Point out that finite Galois extensions of local fields are always solvable, so that decomposition groups are always solvable. More precisely, the decomposition group is the semidirect product of the inertia group by the Galois group of the residue field extension, the inertia group is the semidirect product of the "wild inertia group" by a cyclic group, and the wild inertia group is a p-group, where p is the characteristic of the residue field.

7. If time permits, mention how this all can be interpreted for a non-Galois extension. Namely, if L/K is not Galois, let N/K be its normal closure, put $G = \mathrm{Gal}(N/K)$, and put $H = \mathrm{Gal}(N/L)$. Now for $\mathfrak{b}$ a prime of N above $\mathfrak{p}$, the double cosets $H\backslash G/G_{\mathfrak{b}}$ correspond to the primes of L above $\mathfrak{b}$, and one can relate the splitting of $\mathfrak{p}$ to this decomposition: for examples, see the homework.