Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Lectures 32-33: Different and Discriminant
November 19-21, 2001

Reference. Neukirch, Section III.2.

Outline of lectures.

Unless otherwise specified, L/K will be a finite extension either of number fields or of finite extensions of $\mathbb{Q}_p$.

1. Define the *relative norm* $\mathrm{Norm}_{L/K}(\mathfrak{a})$ of a fractional ideal $\mathfrak{a}$ of L as the fractional ideal of K generated by $\mathrm{Norm}_{L/K}(\alpha)$ for all $\alpha \in \mathfrak{a}$. Notice that for L/K an extension of number fields, the relative norm can be computed locally:
$$\mathrm{Norm}_{L/K}(\mathfrak{a}) = \prod_{\mathfrak{p}} \prod_{\mathfrak{q}} \mathrm{Norm}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}}(\mathfrak{a}) = \prod_{\mathfrak{p}} \prod_{\mathfrak{q}} \mathfrak{p}^{f(\mathfrak{q}/\mathfrak{p})v_{\mathfrak{q}}(\mathfrak{a})}.$$

2. For L/K any finite extension of fields, the *trace pairing* $\langle \cdot, \cdot \rangle : L \times L \to K$ is given by $\langle x, y \rangle = \mathrm{Trace}_{L/K}(xy)$. The trace pairing is nondegenerate (i.e., $\langle x, y \rangle = 0$ for all y implies $x = 0$) if and only if L/K is separable.

3. Notice that the trace pairing takes $\mathfrak{o}_L \times \mathfrak{o}_L$ into $\mathfrak{o}_K$. (The trace of an algebraic integer is the sum of algebraic integers.)

4. Define the *inverse different* $\mathfrak{D}_{L/K}^{-1}$ as the set of $x \in L$ such that $\langle x, y \rangle \in \mathfrak{o}_K$ for all $y \in \mathfrak{o}_L$. Note that this is a fractional ideal of $\mathfrak{o}_L$.

5. Define the *different* $\mathfrak{D}_{L/K}$ as the inverse of the inverse different; this is always an integral ideal. Note that $\mathfrak{D}_{L/K} = \prod_{\mathfrak{p}} \prod_{\mathfrak{q}} \mathfrak{D}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}}$.

6. If M/L is another extension, note that $\mathfrak{D}_{M/K} = \mathfrak{D}_{M/L}\mathfrak{D}_{L/K}$, as an ideal of $\mathfrak{o}_M$. The point is that $\mathrm{Trace}_{L/K}(\mathrm{Trace}_{M/L}(x)) = \mathrm{Trace}_{M/K}(x)$, so the trace pairing from M to K can be computed "all at once" or "in two stages".

7. In case $\mathfrak{o}_L = \mathfrak{o}_K[\alpha]$ for some $\alpha \in \mathfrak{o}_L$, let $P(x)$ be the minimal polynomial of α over $\mathfrak{o}_L$. Show that $\mathfrak{D}_{L/K} = (P'(\alpha))$ by showing that the dual basis of $1, \alpha, \dots, \alpha^{n-1}$ is
$$\frac{b_0}{P'(\alpha)}, \dots, \frac{b_{n-1}}{P'(\alpha)},$$
where $P(x)/(x-\alpha) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$. In the general case, $\mathfrak{D}_{L/K}$ is the ideal generated by $P'(\alpha)$ for all pairs α, P of an element α of $\mathfrak{o}_L$ which generates L over K, and the minimal polynomial P of α.

8. Show that the relative discriminant is the relative norm of the different, by noting that $\mathrm{Disc}(1, \alpha, \dots, \alpha^{n-1}) = \mathrm{Norm}_{L/K}(P'(\alpha))$. (For another approach, see Neukirch, Theorem III.2.9.)

9. Let $\mathfrak{q}$ be a prime of L over $\mathfrak{p}$ and let $e = e(\mathfrak{q}/\mathfrak{p})$ be the ramification index. Show that if $\mathfrak{q}/\mathfrak{p}$ is unramified or tamely ramified, then $v_{\mathfrak{q}}(\mathfrak{D}_{L/K}) = e - 1$; otherwise, $e \leq v_{\mathfrak{q}}(\mathfrak{D}_{L/K}) \leq e - 1 + v_{\mathfrak{q}}(e)$. In particular, $\mathfrak{q}$ is ramified over $\mathfrak{p}$ if and only if $\mathfrak{q}$ divides $\mathfrak{D}_{L/K}$.