Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Problem Set 2: Number Fields and Ideals
Due in class Monday, September 10

Reference. Neukirch, Sections 1.2 and 1.3.

Jargon watch. An ideal of a ring of integers is sometimes called an *integral ideal* to emphasize that it is not just a fractional ideal.

Conjugates, Trace and Norm.

Please submit all of the following problems.

1. (A theorem of Kronecker) Let α be an algebraic integer all of whose conjugates in $\mathbb{C}$ lie in the closed unit disc. Prove that α is a root of unity. (Hint: consider the powers of α.)

2. Let x be an element of a number field K such that $\mathrm{Trace}(x^n)$ is a rational integer for all $n \in \mathbb{N}$. Prove that x is an algebraic integer.

Ideals and Unique Factorization.

Please submit four of the following problems.

1. Find an example to show that a proper subring of the ring of integers of a number field need not have unique factorization of ideals. (Hint: this can already be done in a quadratic number field.)

2. Neukirch exercise I.3.3: Let d be squarefree and p a prime number not dividing $2d$. Let $\mathfrak{o}$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$. Show that $(p) = p\mathfrak{o}$ is a prime ideal of $\mathfrak{o}$ if and only if the congruence $x^2 \equiv d \pmod{p}$ has no solution in rational integers.

3. Show that $\mathbb{Z}[\sqrt{-5}]$ has exactly two ideal classes, the class of principal ideals and the class containing $(2, 1+\sqrt{-5})$. (Hint: suppose $\mathfrak{a}$ is an ideal which is not principal and α is a nonzero element of $\mathfrak{a}$ of minimum norm. Prove that $2\mathfrak{a} \subseteq (\alpha)$.)

4. Neukirch I.3.6: Prove that every ideal of a Dedekind domain can be generated by two elements. (Hint: use Neukirch I.3.5: The quotient of a Dedekind domain by a nonzero ideal is a principal ideal ring.)

5. Neukirch I.3.8: Let $\mathfrak{m}$ be a nonzero integral ideal of the Dedekind domain $\mathfrak{o}$. Show that in every ideal class of $\mathfrak{o}$, there exists an integral ideal prime to $\mathfrak{m}$.

6. Let K be any field and let L be a finite separable extension of the field $K(x)$ of rational functions over K. Prove that the integral closure of $K[x]$ in L is a Dedekind domain. (Hint: imitate the proof of Theorem I.3.1 in Neukirch.)