Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Problem Set 5: Dirichlet's Units Theorem/Valuations
Due in class Monday, October 1

Please submit seven of the following problems.

References. Neukirch, Sections 1.5 and 1.6 (for the units theorem); Section 2.3 (for valuations).

Jargon watch. A number field is *totally real* if all of its embeddings are real. Examples: $\mathbb{Q}$, $\mathbb{Q}(\sqrt{D})$ for $D > 0$, $\mathbb{Q}(\cos \frac{2\pi}{n})$.

Given a real quadratic field and a choice of embedding into $\mathbb{R}$, the *fundamental unit* is the smallest unit strictly greater than 1; together with ± 1, the fundamental unit generates the group of units.

Dirichlet's Units Theorem.

1. Let $D > 1$ be a squarefree integer. Show that the set of solutions of Pell's equation $x^2 - Dy^2 = 1$ in rational integers x, y is nonempty, and all solutions are of the form $x + y\sqrt{D} = \pm(u + v\sqrt{D})^n$ for some integer n, where (u, v) is the solution in positive integers minimizing $u + v\sqrt{D}$. (Hint: relate this to the units theorem in $\mathbb{Q}(\sqrt{D})$.) Optional: prove the same statement for D not a perfect square but no longer squarefree, by reducing to the squarefree case.

2. Neukirch exercise I.7.2: Check the following table of fundamental units ϵ_1 for $\mathbb{Q}(\sqrt{D})$:

D	2	3	5	6	7	10
ϵ_1	$1+\sqrt{2}$	$2+\sqrt{3}$	$(1+\sqrt{5})/2$	$5+2\sqrt{6}$	$8+3\sqrt{7}$	$3+\sqrt{10}$

3. Compute the group of units in the field $\mathbb{Q}(\sqrt[3]{2})$. (Hint: we computed on an earlier homework that the ring of integers is $\mathbb{Z}[\sqrt[3]{2}]$.) Optional: use Magma's built-in function `UnitGroup` to check your answer.

4. Neukirch exercise I.7.4: Let ζ be a primitive p-th root of unity, for p an odd prime. Show that every unit in $\mathbb{Z}[\zeta]$ is equal to a power of ζ times a unit in $\mathbb{Z}[\zeta + \zeta^{-1}]$.

5. Neukirch exercise I.7.5: Let ζ be a primitive m-th root of unity, $m \geq 3$. Show that the numbers $\frac{1-\zeta^k}{1-\zeta}$ for $(k, m) = 1$ are units in the ring of integers of the field $\mathbb{Q}(\zeta)$. The subgroup of the group of units they generate is called the group of *cyclotomic units*.

6. Neukirch exercise I.7.6: Let K be a totally real number field (see above), and let T be a proper nonempty subset of the set of embeddings of K into $\mathbb{R}$. Then there exists a unit ϵ satisfying $0 < \tau\epsilon < 1$ for $\tau \in T$, and $\tau\epsilon > 1$ for $\tau \notin T$. (Hint: apply Minkowski's lattice point theorem to the unit lattice in trace-zero space.)

Valuations.

The following is a fun and surprising application of valuation theory due to Paul Monsky (*American Mathematical Monthly*, 1970): a square cannot be divided into an odd number of nonoverlapping triangles of equal area.

1. Let v be an exponential valuation on K. Prove that v extends to the extension L of K in the following cases:

   (a) $L = K[t]/(P(t))$ for an irreducible polynomial P;

   (b) $L = K(t)$.

   (Hint: for (a), pick $v(t)$ so that the terms of $P(t)$ do not all have distinct valuation. For (b), attempt to set $v(t) = 1$.)

2. Suppose a square is divided into nonoverlapping triangles, and that the vertices of all of the triangles have been colored in red, green and blue. (Note: a side of the square or a triangle may contain vertices of other triangles.) Suppose that none of the sides of the square or the triangles contains vertices of all three colors (including its endpoints), but an odd number of sides of the square have one vertex colored red and one vertex colored blue. Prove that one of the triangles has vertices of three different colors. (Hint: parity.)

3. Suppose the square $[0,1] \times [0,1]$ has been divided into m nonoverlapping triangles of area $1/m$. Let K be the field generated over $\mathbb{Q}$ by the coordinates of the vertices of the triangles. Prove that there exists an extension v to K of the 2-adic exponential valuation of $\mathbb{Q}$.

4. With notation as in the previous problem, color the vertex (x, y) red if $v(x) > 0$ and $v(y) > 0$, blue if $v(x) \leq 0$ and $v(x) \leq v(y)$, and green otherwise. Prove that vertices of three different colors cannot be collinear. (Hint: if you're having trouble dealing with v, first try this assuming $K = \mathbb{Q}$.

5. With notation as in the previous problem, prove that if a triangle has vertices of all three colors, its area has negative valuation. Conclude that m must be even. (Hint: same hint as the previous exercise.)