Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Problem Set 7: Local Fields/Ramification
Due *in Frank Calegari's mailbox* Monday, October 15

This week only: please make any extension requests directly to Frank. His email is `fcale@math.berkeley.edu`.

Please submit seven of the following problems.

Jargon watch. Given a field K and a fixed exponential valuation on K, the *value group* is the image of the exponential valuation in $\mathbb{R}$.

A field with a discrete valuation is called *henselian* if the statement of Hensel's Lemma holds over the field, whether or not the field is complete. For the purposes of this problem set, you may use the fact that a field K is henselian if and only if the following weak form of Hensel's Lemma is true: if $P(x)$ is a polynomial over the valuation ring R of K and α is a simple root of P over the residue field of R, then there is a root of P in R that reduces to α. See Section II.6 for more on henselian fields.

Throughout the set, let ζ_n denote an n-th root of unity.

More on local fields.

1. Determine all of the quadratic extensions of $\mathbb{Q}_p$. (Your answer will have a different form for $p = 2$.)

2. Determine the degree of $\mathbb{Q}_p(\zeta_n)$ over $\mathbb{Q}_p$, for p a prime not dividing n.

3. Let K be the maximal unramified extension of $\mathbb{Q}_p$. Prove that K is henselian. (Hint: reduce to a finite extension of $\mathbb{Q}_p$.)

4. Let R be the ring of power series over $\mathbb{C}$ that converge in some neighborhood of 0. Prove that the fraction field of R is henselian.

5. Neukirch II.6.1 (modified): Let $f(x) = x^n + \sum_{i=0}^{n-1} f_i x^i$ be a monic polynomial of degree n over a local field K, which factors completely over K with roots $r_1, \dots, r_n$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $g(x) = x^n + \sum_{i=0}^{n-1} g_i x^i$ is a monic polynomial of degree n such that $|f_i - g_i| < \delta$, then g has n roots $s_1, \dots, s_n$ in K, which can be labeled so that $|r_i - s_i| < \epsilon$ for $i = 1, \dots, n$. That is, the roots of f vary continuously with the coefficients.

Ramification.

1. Prove that a finite extension of $\mathbb{Q}_p$ has only one unramified extension of any given degree. (Hint: recall that a finite field has only one extension of any given degree.)

2. Show that $\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(\pi)$ for π a $(p-1)$-st root of $-p$. (Hint: both extensions are tamely ramified.)

3. Neukirch II.7.1: The maximal unramified extension of $\mathbb{Q}_p$ is obtained by adjoining all roots of unity of order prime to p.

4. Neukirch II.7.3: Let K be a local field and L/K a totally and tamely ramified extension, and let Γ and Δ be the value groups of K and L, respectively. Show that the subfields of L containing K are in one-to-one correspondence with the subgroups of Δ/Γ.

5. Let $P(x)$ be a monic irreducible polynomial over $\mathbb{Z}_p$. Suppose the discriminant of $P(x)$ is not divisible by p. Prove that the field $\mathbb{Q}_p[x]/(P(x))$ is unramified over $\mathbb{Z}_p$.

6. For p odd, prove that $\mathbb{Q}_p(\zeta_n)$ is ramified over $\mathbb{Q}_p$ if and only if p divides n. Optional: what happens if $p = 2$? (Hint: see the end of Section II.7.)