Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Problem Set 8: Ramification and Splitting
Due in class Monday, October 29

Please submit six of the following problems.

Extension of Valuations.

1. Neukirch exercise II.8.3: How many extensions to $\mathbb{Q}(\sqrt[n]{2})$ does the archimedean absolute value $|\cdot|$ of $\mathbb{Q}$ admit?

Ramification and Splitting.

1. Neukirch I.8.3: If a prime ideal $\mathfrak{p}$ of a number field K is totally split in the extensions L/K and L'/K, then it is also totally split in the compositum LL'/K.
2. Write a Magma program to determine the splitting behavior (i.e., the number of prime factors and the corresponding pairs (e, f)) of primes of $\mathbb{Q}$ in the ideal $\mathbb{Q}(\sqrt[3]{2})$. (Hint: relate the splitting to the factorization of the polynomial $x^3 - 2$ modulo p.) Optional: compute some examples and look for patterns in the data; it may help to separate primes based on $p \pmod 3$.

Interlude on Quadratic Reciprocity.

1. For $n = \prod_i p_i^{e_i}$ odd and coprime to a, define the Legendre symbol $\left(\frac{a}{n}\right)$ as $\prod_i \left(\frac{a}{p_i}\right)^{e_i}$. Formulate a version of quadratic reciprocity (Theorem 8.6) for this generalized Legendre symbol, and use this to describe a version of the Euclidean algorithm to compute Legendre symbols *without* factoring into primes.
2. Neukirch II.8.8: Let $a_n = \frac{\epsilon^n - \epsilon'^n}{\sqrt{5}}$, where $\epsilon = \frac{1+\sqrt{5}}{2}$, $\epsilon' = \frac{1-\sqrt{5}}{2}$, so that a_n is the n-th Fibonacci number. If p is a prime number other than 2 or 5, prove that $a_p \equiv \left(\frac{p}{5}\right) \pmod p$.

The Relative Discriminant.

1. Neukirch defines the relative discriminant of an extension L/K of number fields as the ideal generated by $\mathrm{Disc}(\alpha_1, \dots, \alpha_n)$ for all choices $\alpha_1, \dots, \alpha_n$ of a basis of L/K consisting of elements of $\mathfrak{o}_L$. Show that this coincides with the definition I gave in class. (Hint: you might want to peek ahead to Section III.2.)
2. If L/K is an extension of number fields, prove that the ideal $(\mathrm{Disc}(L)/\mathrm{Disc}(K))$ (the quotient of the absolute discriminants) is equal to the absolute norm of the relative discriminant $\mathrm{Disc}(L/K)$. (Hint: use the definition of the relative discriminant from class.)

3. Prove that if $P(x)$ is an irreducible monic polynomial over $\mathbb{Q}$ whose discriminant D is squarefree, then $\mathbb{Q}[x]/(P(x))$ is unramified over $\mathbb{Q}(\sqrt{D})$.

4. Compute the relative discriminant $\mathrm{Disc}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, using the fact that the valuation ring of $\mathbb{Q}_p(\zeta_p)$ equals $\mathbb{Z}_p[\zeta_p]$, proved in class (and in Section II.7 of Neukirch). Use this result to prove that the ring of integers of $\mathbb{Q}(\zeta_p)$ equals $\mathbb{Z}[\zeta_p]$. Optional: peek ahead to Section I.10 and compare the proof given there.