Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Problem Set 9: Ramification in Galois Extensions
Due in class Monday, November 5

Please do six of the following problems, including the first problem.

Leftover Problem about Ramification.

1. Let K be a finite extension of $\mathbb{Q}_p$ and L/K be a tamely ramified (but not necessarily totally ramified) extension. Compute the discriminant $\mathrm{Disc}(L/K)$ in terms of $e(L/K)$ and $f(L/K)$. (The point: the discriminant only gets really nasty in the wildly ramified case.)

Ramification in Galois Extensions.

1. Neukirch exercise I.9.1: if L/K is a Galois extension of number fields with noncyclic Galois group, then there are at most finitely many totally nonsplit prime ideals of K (i.e., primes of $\mathfrak{o}_K$ with only one prime of $\mathfrak{o}_L$ over them). Also, use this argument to explicitly construct an irreducible polynomial over $\mathbb{Z}$ which is not irreducible modulo any prime.

2. Neukirch exercise I.9.2 (corrected): if L/K is a Galois extension of number fields and $\mathfrak{b}$ is a prime ideal which is unramified over L (i.e., for $\mathfrak{p} = \mathfrak{b} \cap K$, $e(\mathfrak{b}/\mathfrak{p}) = 1$), there exists a unique automorphism $\phi_{\mathfrak{b}}$ of L over K such that $\phi_{\mathfrak{b}}(a) \equiv a^q \pmod{\mathfrak{b}}$, where $q = \mathrm{Norm}(\mathfrak{p})$. It is called the *Frobenius automorphism* of $\mathfrak{b}$; observe also that $\phi_{\mathfrak{b}}$ generates the decomposition group of $\mathfrak{b}$ in this case.

3. Neukirch I.9.3: let L/K be an extension of prime degree p with solvable normal closure. If the unramified prime ideal $\mathfrak{p}$ in L has two prime factors $\mathfrak{b}$ and $\mathfrak{b}'$ of inertia degree 1, then it is totally split. (Hint from Neukirch: use a theorem of Galois, that if G is a transitive solvable permutation group of prime degree p, then there is no nontrivial permutation of G that fixes two distinct letters.)

4. Neukirch I.9.4: Let L/K be a finite, but not necessarily Galois, extension of number fields and N/K the normal closure of L/K. Show that a prime ideal $\mathfrak{p}$ of K is totally split in L if and only if it is totally split in N. (Hint from Neukirch: use the double coset decomposition $H\backslash G/G_{\mathfrak{b}}$, where $G = G(N/K)$, $H = G(N/L)$, and $G_{\mathfrak{b}}$ is the decomposition groups of a prime ideal $\mathfrak{b}$ over $\mathfrak{p}$.)

5. Prove that the previous exercise remains true if "totally split" is replaced by "unramified" or by "tamely ramified". Optional: describe more generally how to compute the ramification and inertia degrees of the primes of L above $\mathfrak{p}$ in terms of the double coset decomposition.

6. Let $P(x)$ be a polynomial of degree n over $\mathfrak{o}_K$ for some number field K such that modulo some prime $\mathfrak{p}$, $P(x)$ is irreducible, while modulo some other prime $\mathfrak{q}$, $P(x)$ factors as an irreducible quadratic times pairwise coprime linear factors. Prove that the Galois group of the normal closure of $K[x]/(P(x))$ is the full symmetric group S_n. (Hint: use Frobenius automorphisms.)

7. Formulate a sensible definition of "the decomposition/inertia group of an archimedean embedding of a number field". As a test, your definition should allow the proof of the preceding problem to pass through unchanged to prove the following: if $P(x)$ is a polynomial over $\mathbb{Z}$ which is irreducible modulo some prime p and which has exactly two nonreal roots, then the Galois group of the normal closure of $\mathbb{Q}[x]/(P(x))$ is the full symmetric group S_n.