Math 254A, UC Berkeley, Fall 2001 (Kedlaya)
Addendum on the Relative Discriminant
October 31, 2001

A lot of you had trouble with the homework problems on the relative discriminant last week. One big problem was typos on the problem set. The correct statement of problem 2 was: If L/K is a number field, prove that the ideal $(\mathrm{Disc}(L)/\mathrm{Disc}(K)^{[L:K]})$ is the ideal generated by the absolute norm of the relative discriminant $\mathrm{Disc}(L/K)$. (I forgot the factor of $[L:K]$ in the exponent. Mea maxima culpa.)

One approach to computing this is the one taken by Neukirch in Section 3.2, using the different. I had in mind a more direct argument, something like the following.

It is true (but I didn't say this explicitly in class, and should have) that trace, norm and the discriminant can be computed two different ways for local fields just like for number fields. Namely, if L/K is an extension of finite extensions of $\mathbb{Q}_p$ of degree n, there will be exactly n embeddings $\tau_1, \dots, \tau_n$ of L into $\overline{K}$ (that much is true for any separable extension), and we can write

$$\mathrm{Trace}_{L/K}(\alpha) = \sum_i \tau_i(\alpha), \quad \mathrm{Norm}_{L/K}(\alpha) = \prod_i \tau_i(\alpha), \quad \mathrm{Disc}(\alpha_1, \dots, \alpha_n) = (\det(\tau_i(\alpha_j))_{i,j=1}^n)^2;$$

the equivalence of these definitions with the original definitions works the same way as in the number field case.

Using problem 1 and the relationship between the absolute discriminant and Neukirch's form of the relative discriminant, problem 2 reduces to the following: if M/L and L/K are extensions of finite extensions of $\mathbb{Q}_p$, then

$$\mathrm{Disc}(M/K) = \mathrm{Disc}(L/K)^{[M:K]} \,\mathrm{Norm}_{L/K} \mathrm{Disc}(M/L), \tag{1}$$

where $\mathrm{Norm}_{L/K}$ sends an ideal of $\mathfrak{o}_L$ generated by α to the ideal $(\mathrm{Norm}_{L/K}(\alpha))$. Here's one way to prove this.

Put $m = [M : L]$ and $n = [L : K]$, choose a basis $\alpha_1, \dots, \alpha_m$ of $\mathfrak{o}_M$ over $\mathfrak{o}_L$ and a basis $\beta_1, \dots, \beta_m$ of $\mathfrak{o}_L$ over $\mathfrak{o}_K$, and choose an isomorphism $f : \overline{L} \to \overline{K}$ of fields once and for all. (Note: $\overline{L}$ is a field together with a fixed map of L into it, so I'll push elements of L into it without comment. Likewise, $\overline{K}$ is a field together with a fixed map of K into it.)

Let $\tau_1, \dots, \tau_m$ be the embeddings of M into $\overline{L}$ over L, and let $\sigma_1, \dots, \sigma_n$ be the embeddings of L into $\overline{K}$ over K. Choose automorphisms $\rho_1, \dots, \rho_n$ of $\overline{K}$ such that $\rho_i \circ f = \sigma_i$ for each i. Now $\rho_i \circ \tau_j$ gives all of the embeddings of M into $\overline{K}$ over K, so using the basis $\alpha_k \beta_l$ of $\mathfrak{o}_M$ over $\mathfrak{o}_K$ (for $k = 1, \dots, m$ and $l = 1, \dots, n$), I can write $\mathrm{Disc}(M/K) = (\det C)^2$, where the matrix C has rows and columns indexed by ordered pairs and is given by

$$C_{(i,j)(k,l)} = \rho_i(\tau_j((\alpha_k \beta_l))) = \rho_i(\tau_j(\alpha_k)) \cdot \rho_i(\beta_l)). \tag{2}$$

(The τ_j dropped out in the second term because we have a fixed identification of L as a subfield of $\overline{L}$, and τ_j is L-linear.)

I'd like to factor this determinant using the identity $\det(A \otimes B) = (\det A)^n (\det B)^m$ for the determinant of the tensor product of an $m \times m$ matrix A and an $n \times n$ matrix B. (The *tensor product* of A and B is the matrix with rows and columns indexed by pairs (i,j) and $(A \otimes B)_{(i,j)(k,l)} = A_i k B_{jl}$. Why it's called that is left to the reader.) Unfortunately, I haven't quite got that yet, but I can modify the determinant in question to bring this about.

For ease of notation, let T be the matrix with $T_{ij} = \tau_i(\alpha_j)$, so that $C_{(i,j)(k,l)} = \rho_i(M_{jk})\rho_i(\beta_l)$. Now define the matrix D by

$$D_{(i,j)(k,l)} = \begin{cases} (T\rho_i(T)^{-1})_{jl} & i = k \\ 0 & i \neq k. \end{cases}$$

We have

$$\begin{aligned} (DC)_{(i,j)(k,l)} &= \sum_{(r,s)} D_{(i,j)(r,s)} C_{(r,s)(k,l)} \\ &= \sum_s D_{(i,j)(i,s)} C_{(i,s)(k,l)} \\ &= \sum_s (T\rho_i(T)^{-1})_{js} \rho_i(T)_{jk} \rho_i(\beta_l) \\ &= T_{jk}\rho_i(\beta_l). \end{aligned}$$

That is, DC is the tensor product of the matrices M and N, where $N_{il} = \rho_i(\beta_l)$. Since D is a block matrix, we can compute its determinant as

$$\det(D) = \prod_i \frac{\det T}{\det \rho_i(T)}.$$

We conclude

$$\begin{aligned} \det C &= (\det T^{-m} \prod_i \det \rho_i(T)) \det T^m \det N^m) \\ &= (\det N)^m \prod_i \rho_i(\det T) \\ &= (\det N)^m \operatorname{Norm}_{L/K}(\det T). \end{aligned}$$

To conclude, observe that $(f^{-1}(\det T))^2 = (\operatorname{Disc}(M/L))$ and $(\det N)^2 = (\operatorname{Disc}(L/K))$.