ZETA FUNCTIONS AND L-SERIES FOR A NUMBER FIELD

CARL MILLER

1. Introduction

Every number field K has an associated zeta function $\zeta_K(z)$, whose analytic properties are closely tied to number theoretic information about K. Zeta functions are also a subset of a larger class of analytic functions called L-series, which come from one-dimensional representations of subgroups of the ideal class group. These functions are useful for proving many results in class field theory.

We are concerned specifically with generalized Dirichlet L-series, which are associated to the ray class group of K for a chosen modulus $\mathfrak{m}$. We will define an L-series and prove convergence for $\mathrm{Re}(z) > 1$. Since we are primarily interested in the values of the L-series near $z = 1$, we will need to show that the L-series has an analytic continuation to a neighborhood of 1. Our proof of the continuation relies on an important theorem about the distribution of norms in a generalized ideal class. In Section 6 we will give a summary proof of this theorem, referring the reader to Lang for some technical lemmas. Then, as an example application, we use L-series to prove a result in class field theory known as the Universal Norm Index Inequality.

2. Notation and Definitions

Throughout, let $K/\mathbb{Q}$ be a number field, let A denote its ring of integers, and let $\mathbf{I}$ and $\mathbf{P}$ denote the group of fractional ideals of K and principal fractional ideals of K, respectively. Let $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$ be a formal product of places of K, with $\mathfrak{m}_0$ the finite part and $\mathfrak{m}_\infty$ the infinite part. If $v \mid \mathfrak{m}_\infty$ is a valuation, let $\sigma_v : K \to k_v$ denote the corresponding embedding, with $k_v = \mathbb{R}$ or $\mathbb{C}$. For convenience we will write $x \equiv 1 \mod \mathfrak{m}$ to mean that

$$x \equiv 1 \mod \mathfrak{p}^m$$

for all $\mathfrak{p}^m \mid \mathfrak{m}_0$, and

$$\sigma_v(x) > 0$$

for all real $v \mid \mathfrak{m}_\infty$. Let $\mathbf{I}^\mathfrak{m}$ denote the group of fractional ideals that are coprime to $\mathfrak{m}_0$, let $\mathbf{P}^\mathfrak{m} \subseteq \mathbf{I}^\mathfrak{m}$ denote the subgroup generated by principal ideals (x) with $x \equiv 1 \mod \mathfrak{m}$, and let $\mathrm{Cl}_\mathfrak{m}(K) = \mathbf{I}^\mathfrak{m}/\mathbf{P}^\mathfrak{m}$ denote the ray class group of $\mathfrak{m}$. If $J \subseteq A$ is an ideal, let $\mathbf{N}(J)$ denote the norm of J.

3. Dirichlet Characters

Let $\Xi_\mathfrak{m}$ be the group of complex characters $\chi : \mathrm{Cl}_\mathfrak{m}(K) \to \mathbb{C}^*$. We note a few properties of this group. Recall that the ray class group is always finite (see [2], pp. 112).

Proposition 3.1. *The order of $\Xi_\mathfrak{m}$ is the same as the order of the group $\mathrm{Cl}_\mathfrak{m}(K)$.*

Proof. Write $\mathrm{Cl}_{\mathfrak{m}}(K)$ as a direct product of finite cyclic groups,

$$\mathrm{Cl}_{\mathfrak{m}}(K) \cong \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_r\mathbb{Z}$$

and let $g_i \in \mathrm{Cl}_{\mathfrak{m}}(K)$ be their respective generators. A character χ is determined by the independent choice of mappings of each g_i to an n_i-th root of unity. There are $n_1 n_2 \cdots n_r = |\mathrm{Cl}_{\mathfrak{m}}(K)|$ possible choices. □

Proposition 3.2. *Let $\chi_0 \in \Xi_{\mathfrak{m}}$ denote the trivial character. Any character χ : $\mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbb{C}^*$ satisfies*

$$\sum_{g \in \mathrm{Cl}_{\mathfrak{m}}(K)} \chi(g) = \begin{cases} |\mathrm{Cl}_{\mathfrak{m}}(K)| & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

Proof. Note that for any $h \in \mathrm{Cl}_{\mathfrak{m}}(K)$,

$$\sum_{g \in \mathrm{Cl}_{\mathfrak{m}}(K)} \chi(g) = \sum_{gh \in \mathrm{Cl}_{\mathfrak{m}}(K)} \chi(gh) = \chi(h) \sum_{g \in \mathrm{Cl}_{\mathfrak{m}}(K)} \chi(g).$$

So either $\sum_{g \in \mathrm{Cl}_{\mathfrak{m}}(K)} \chi(g) = 0$, or $\chi(h) = 1$ for all $h \in \mathrm{Cl}_{\mathfrak{m}}(K)$, and hence $\sum_{g \in \mathrm{Cl}_{\mathfrak{m}}(K)} = |\mathrm{Cl}_{\mathfrak{m}}(K)|$, as desired. □

By a similar trick, we have

Proposition 3.3. *For any ideal class $g \in \mathrm{Cl}_{\mathfrak{m}}(K)$,*

$$\sum_{\chi \in \Xi_{\mathfrak{m}}} \chi(g) = \begin{cases} |\Xi_{\mathfrak{m}}| = |\mathrm{Cl}_{\mathfrak{m}}(K)| & \text{if } g = \boldsymbol{P}^{\mathfrak{m}}, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

4. Zeta Functions and L-series

For a generalized Dirichlet character $\chi : \mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbb{C}^*$, the L-series $L_{\mathfrak{m}}(z, \chi)$ is given by

$$L_{\mathfrak{m}}(z, \chi) = \sum_{(J, \mathfrak{m})=1} \frac{\chi(J)}{\mathbf{N}(J)^z}$$

where the sum is over integral ideals J that are coprime to $\mathfrak{m}$. Note that if $\mathfrak{m} = 1$ and $\chi : \mathrm{Cl}(K) \to \mathbb{C}^*$ is trivial, we obtain the zeta function of the number field K:

$$\zeta_K(z) = \sum_{J \subseteq A} \frac{1}{\mathbf{N}(J)^z}.$$

It will be useful when working with L-series to refer to the zeta function of an ideal class $\mathfrak{J} \in \mathrm{Cl}_{\mathfrak{m}}(K)$, given by

$$\zeta(z, \mathfrak{J}) = \sum_{J \in \mathfrak{J}} \frac{1}{\mathbf{N}(J)^z}.$$

Note that

$$L_{\mathfrak{m}}(z, \chi) = \sum_{\mathfrak{J} \in \mathrm{Cl}_{\mathfrak{m}}(K)} \chi(\mathfrak{J}) \zeta(z, \mathfrak{J}).$$

We show first that these series converge when $\mathrm{Re}(z) > 1$. Note that for p prime, the number of ideals $\mathfrak{p}$ satisfying $\mathbf{N}(\mathfrak{p}) = p$ is less than or equal to $d = [K : \mathbb{Q}]$, and secondly, the number of prime factors (counted with multiplicity) for a given n is less than or equal to $\log_2 n$. Combining these facts shows that the number of ideals

J satisfying $\mathbf{N}(J) = n$ is less than or equal to $(\log_2 n)^d$. Thus all the series above are dominated (in absolute value) by

$$\sum_{n=1}^{\infty} \frac{(\log_2 n)^d}{n^{\mathrm{Re}(z)}},$$

which converges provided that $\mathrm{Re}(z) > 1$.

The following product form of the L-series will be useful in later proofs.

Proposition 4.1. *For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1$,*

$$L_{\mathfrak{m}}(z, \chi) = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \frac{1}{1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^z}}.$$

Proof. Taking the formal logarithm of the series gives

$$\sum_{\mathfrak{p} \nmid \mathfrak{m}} \sum_{n=1}^{\infty} \frac{\chi(\mathfrak{p})^n}{n\mathbf{N}(\mathfrak{p})^{nz}}$$

which is dominated by

$$\sum_{p \in \mathbb{N}} \sum_{n=1}^{\infty} \frac{d}{np^{nz}} < \sum_{m \in \mathbb{N}} \frac{d}{m^z},$$

and this converges for $\mathrm{Re}(z) > 1$. Thus we may safely expand and multiply out the series:

$$\prod_{\mathfrak{p} \nmid \mathfrak{m}} \frac{1}{1 - \frac{\chi(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^z}} = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \left(1 + \frac{\chi(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^z} + \frac{\chi(\mathfrak{p})^2}{\mathbf{N}(\mathfrak{p})^{2z}} + \cdots\right)$$

The function $\chi(\cdot)/\mathbf{N}(\cdot)^z$ is multiplicative, thus

$$\prod_{\mathfrak{p} \nmid \mathfrak{m}} \left(1 + \frac{\chi(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^z} + \frac{\chi(\mathfrak{p})^2}{\mathbf{N}(\mathfrak{p})^{2z}} + \cdots\right) = \sum_{(J, \mathfrak{m})=1} \frac{\chi(J)}{\mathbf{N}(J)^z}$$

as desired. □

Already we have seen a connection between the L-series and the number theoretic properties of K: the multiplicative expression for $L_{\mathfrak{m}}(z, \chi)$ is immediately equivalent to unique factorization for ideals in A.

5. The Riemann Zeta Function

We have seen that the series for $L_{\mathfrak{m}}(z, \chi)$ converges to an analytic function for $\mathrm{Re}(z) > 1$. However since we are interested in the values of $L_{\mathfrak{m}}(z, \chi)$ near $z = 1$, we need to prove the existence of a larger analytic continuation. We begin by considering the Riemann zeta function,

$$\zeta_{\mathbb{Q}}(z) = \sum_{p \in \mathbb{N}} \frac{1}{p^z}.$$

The convergence theorem for the Riemann zeta function will be useful as an intermediate step in proving a similar theorem for general L-series.

Theorem 5.1. *There exists an analytic continuation of $\zeta_{\mathbb{Q}}(z)$ to the domain $\{z : \mathrm{Re}(z) > 0, z \neq 1\}$.*

Proof. Consider the alternating series

$$\zeta_2(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \dots.$$

Let $\sigma = \mathrm{Re}(z)$; suppose $\sigma > 0$. Then

$$\begin{aligned} |\zeta_2(z)| &= \left| z \int_1^2 \frac{ds}{s^{z+1}} + z \int_3^4 \frac{ds}{s^{z+1}} + z \int_5^6 \frac{ds}{s^{z+1}} + \dots \right| \\ &\leq \frac{|z|}{1^{\sigma+1}} + \frac{|z|}{3^{\sigma+1}} + \frac{|z|}{5^{\sigma+1}} + \dots \end{aligned}$$

which is convergent. Thus $\zeta_2(z)$ is analytic on $\{z : \mathrm{Re}(z) > 0\}$. Now note that

$$\zeta_{\mathbb{Q}}(z) - \left(\frac{2}{2^z}\right) \zeta_{\mathbb{Q}}(z) = \zeta_2(z),$$

for $\mathrm{Re}(z) > 1$, hence the formula

$$\zeta_{\mathbb{Q}}(z) = \left(1 - \frac{1}{2^{z-1}}\right)^{-1} \zeta_2(z)$$

gives an analytic continuation of $\zeta_{\mathbb{Q}}(z)$ to the region $\{z : \mathrm{Re}(z) > 0\}$ minus the zeroes of the function $\left(1 - 1/2^{z-1}\right)$. We can take care of all but one of these exceptional points by considering another alternating zeta series,

$$\zeta_3(z) = \sum_{n=1}^{\infty} \left(\frac{1}{(3n+1)^z} + \frac{1}{(3n+2)^z} - \frac{2}{(3n+3)^z} \right).$$

By similar reasoning as above, $\zeta_3(z)$ converges for $\mathrm{Re}(z) > 0$, and

$$\zeta_{\mathbb{Q}}(z) = \left(1 - \frac{1}{3^{z-1}}\right)^{-1} \zeta_3(z)$$

Thus we have an analytic continuation of $\zeta_{\mathbb{Q}}(z)$ to $\{z : \mathrm{Re}(z) > 0\}$ excluding the zeroes of $\left(1 - 1/3^{z-1}\right)$. Now suppose z is a zero of both $\left(1 - 1/2^{z-1}\right)$ and $\left(1 - 1/3^{z-1}\right)$. Then $2^{z-1} = 3^{z-1} = 1$, hence $z = 1 + \theta i$ and

$$\theta \log 2 \equiv \theta \log 3 \equiv 0 \mod 2\pi$$

But $\log 2 / \log 3$ is irrational, so $\theta = 0$. Thus $z = 1$ is the only excluded point. □

Proposition 5.2. *The zeta function $\zeta_{\mathbb{Q}}(z)$ has a simple pole at $z = 1$ with residue 1.*

Proof. Using the expression from the last proof,

$$\zeta_{\mathbb{Q}}(z) = \left(1 - \frac{1}{2^{z-1}}\right)^{-1} \zeta_2(z).$$

Now $\zeta_2(1)$ is the power series expression for $\log 2$, while the residue of $\left(1 - 1/2^{z-1}\right)^{-1}$ at 1 is $1/\log 2$. Thus $\mathrm{Res}_{z=1}\, \zeta_{\mathbb{Q}}(z) = 1$. □

In fact any zeta function may be extended to $\mathbb{C} - \{1\}$, but it is not necessary for us to prove that here (see [3], Ch. VII). All "nontrivial" zeroes of the Riemann zeta function lie in the strip $\{z : 0 \leq z \leq 1\} \subseteq \mathbb{C}$, so what we have proved above is sufficient to state the famous Riemann hypothesis.

Conjecture 5.3. All nontrivial zeroes of $\zeta_{\mathbb{Q}}(z)$ lie on the line $\{z : \mathrm{Re}(z) = 1/2\}$.

6. The Distribution of Norms in an Ideal Class

Let $\mathfrak{J} \in \mathrm{Cl}_{\mathfrak{m}}(K)$ be an ideal class. In order to evaluate the zeta function $\zeta(z, \mathfrak{J})$, we need an asymptotic estimate for the function

$$j(\mathfrak{J}, t) = |\{J \in \mathfrak{J} : J \subseteq A, \mathbf{N}(J) \leq t\}| .$$

Choose an (integral) ideal B from the inverse class $\mathfrak{J}^{-1}$. The one-to-one map $J \mapsto JB$ takes an integral ideal in $\mathfrak{J}$ to a principal ideal; the image of this map is precisely the set of ideals in $P^{\mathfrak{m}}$ that are generated by elements of B. Also, $\mathbf{N}(J) \leq t$ if and only if $\mathbf{N}(JB) \leq t\mathbf{N}(B)$. So it suffices to count the number of ideals $(x) \in P^{\mathfrak{m}}$ such that $x \in B$ and $\mathbf{N}(x) \leq t\mathbf{N}(B)$.

To put this another way:

Lemma 6.1. *The function $j(\mathfrak{J}, t)$ is equal to the number of elements $x \in B$ satisfying*

$$x \equiv 1 \mod \mathfrak{m} \qquad \text{and} \qquad \boldsymbol{N}(x) \leq t\boldsymbol{N}(B)$$

modulo the unit group $U^{\mathfrak{m}}$. $\square$

We will describe a geometric interpretation for the set of such elements. Let S_∞ denote the set of infinite primes of K. The embeddings $\{\sigma_v\}_{v \in S_\infty}$ give an embedding

$$K \to \prod_{v \in S_\infty} k_v \cong \mathbb{R}^n.$$

The vector space $\prod_{v \in S_\infty} k_v$ has a norm

$$\mathbf{N}(\mathbf{x}) = \prod_{v \in S_\infty} v(x_v)^{N_v}$$

(here N_v denotes the multiplicity of v in $\mathfrak{m}_\infty$). Thus we define the unit ball $B^1 \subseteq \prod_{v \in S_\infty} k_v$. Note that the image of an ideal under this map is a lattice.

We wish to make use of the following idea: if a region $D \subseteq \mathbb{R}^n$ is sufficiently nice, then we can estimate its volume to an arbitrary degree of accuracy by counting lattice points inside D for sufficiently fine lattices. This is made precise in the theorem below.

Theorem 6.2. *Suppose $D \subseteq \mathbb{R}^n$ is a subset whose boundary is $(n-1)$-Lipshitz parametrizable, and suppose $L \subseteq \mathbb{R}^n$ is a lattice with fundamental parallelotope P. Let $\lambda(t, D, L)$ be the number of lattice points in tD. Then*

$$\lambda(t, D, L) = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(P)} t^n + O(t^{n-1}).$$

Proof. See [1], pp. 128. $\square$

We will show that $j(\mathfrak{J}, t)$ is equal to $\lambda(t, D, L)$ for appropriately chosen D and L. First note that the group of units $U^{\mathfrak{m}}$ acts on $\prod_{v \in S_\infty} k_v$ via $u(\mathbf{x}) = (q(u) \cdot \mathbf{x})$. This action becomes fixed-point free if we restrict to the subset

$$Q(S_\infty) = \{(q_v)_{v \in S_\infty} : q_v \neq 0 \text{ for all } v, \text{ and } q_v > 0 \text{ for real } v \mid \mathfrak{m}_\infty\}.$$

Let $V \subseteq U^{\mathfrak{m}}$ be a maximal free subgroup.

Lemma 6.3. *There exists a fundamental domain $F \subseteq \mathbb{R}^n$ for the action of V on $Q(S_\infty)$ such that $tF = F$ for all $t > 0$. Also, $F \cap B^1$ has $(N-1)$-parametrizable boundary.*

Proof. See [1], pp. 131. $\square$

Now let $D = F \cap B^1$ and let $L \subseteq \mathbb{R}^n$ be the image of the $B\mathfrak{m}_0$.

Choose (by the Chinese Remainder Theorem) an element $y \in B$ such that $y \equiv 1 \mod \mathfrak{m}_0$. Consider the translated lattice $y + L$. Every element of $(y+L) \cap F$ is the image of an element $x \in B$ such that $(x) \in \mathbf{P}^{\mathfrak{m}}$ (recall Lemma 6.1). Conversely, every such principal ideal (x) has exactly $|U^{\mathfrak{m}}/V|$ representatives in $(y+L) \cap F$, since F is a fundamental domain for the action $V \subseteq U^{\mathfrak{m}}$. Let $w_{\mathfrak{m}} = |U^{\mathfrak{m}}/V|$ (this is the number of roots of unity of K, which is finite). We have

$$\begin{aligned} w_{\mathfrak{m}} j(\mathfrak{J}, t^n) &= |\{\mathbf{x} \in \mathbb{R}^n : x \in F, \mathbf{N}(x) \leq \mathbf{N}(B) t^n\}| \\ &= \lambda(t\mathbf{N}(B), D, y+L). \end{aligned}$$

The translation by y is asymptotically irrelevant; by Theorem 6.2,

$$w_{\mathfrak{m}} j(\mathfrak{J}, t^n) = \frac{\mathbf{N}(B)\operatorname{Vol}(D)}{\operatorname{Vol}(P)} t^n + O(t^{n-1}).$$

where P denotes the fundamental parallelotope of L. Therefore,

$$j(\mathfrak{J}, t) = \frac{\mathbf{N}(B)\operatorname{Vol}(D)}{w_{\mathfrak{m}}\operatorname{Vol}(P)} t + O(t^{1-1/n}).$$

Now, the volume of the fundamental parallelotope corresponding to B is linearly related to the norm of B (see [1], pp. 115):

$$\frac{\mathbf{N}(B)}{\operatorname{Vol}(P)} = \frac{2^r}{\sqrt{|D_K|}}$$

where r is the number of complex primes of K and D_K is the discriminant of K. Thus let

$$\rho_{\mathfrak{m}} = \frac{\mathbf{N}(B)\operatorname{Vol}(D)}{w_{\mathfrak{m}}\operatorname{Vol}(P)} = \frac{2^r \operatorname{Vol}(D)}{w_{\mathfrak{m}}\sqrt{|D_K|}}.$$

We have proven

Theorem 6.4. *There exist constants $\rho_{\mathfrak{m}}$ and $n > 1$ such that for any $\mathfrak{J} \in \mathrm{Cl}_{\mathfrak{m}}(K)$, the function $j(\mathfrak{J}, t) = |\{J \in \mathfrak{J} : \mathbf{N}(J) \leq t\}|$ satisfies*

$$j(\mathfrak{J}, t) = \rho_m t + O(t^{1-1/n}).$$

Thus we see that the distribution of the norm function on an ideal class is asymptotically linear, and moreover, the linear coefficient is the same for any ideal class in $\mathrm{Cl}_m(K)$.

We note that there is a nice formula for $\operatorname{Vol}(D)$ in terms of the invariants of the number field K (but we won't need to prove that here). This leads to a formula for the residue of $\zeta_K(z)$ at 1 called the Class Number Formula (see [3], pp. 467).

7. L-series Convergence

Consider the ideal class zeta function,

$$\zeta(z, \mathfrak{J}) = \sum_{J \in \mathfrak{J}} \frac{1}{\mathbf{N}(J)^z}.$$

Let $a_m = |\{J \in \mathfrak{J} : \mathbf{N}(J) = m\}|$. We have

$$\zeta(z, \mathfrak{J}) = \sum_{m=1}^{\infty} \frac{a_m}{m^z},$$

and by Theorem 6.4,

$$\sum_{m=1}^{t} a_m = \rho_{\mathfrak{m}} t + O(t^{1-1/n}).$$

We will demonstrate an analytic continuation of $\zeta(z, \mathfrak{J})$ via comparison with the Riemann zeta function. Let $s_t = \sum_{m=1}^{t}(a_m - \rho_{\mathfrak{m}})$. Observe:

$$\begin{aligned}
\left|\sum_{m=1}^{t} \frac{a_m - \rho_{\mathfrak{m}}}{m^z}\right| &= \left|\sum_{m=1}^{t} \frac{s_m}{m^z} - \sum_{m=0}^{t-1} \frac{s_m}{(m+1)^z}\right| \\
&= \left|\frac{s_t}{t^z} + \sum_{m=1}^{t-1} s_m \left(\frac{1}{m^z} - \frac{1}{(m+1)^z}\right)\right| \\
&\leq \left|\frac{s_t}{t^z}\right| + \left|\sum_{m=1}^{t-1} s_m\, z \int_m^{m+1} \frac{ds}{s^{z+1}}\right| \\
&\leq \left|\frac{s_t}{t^z}\right| + \sum_{m=1}^{t-1} \frac{|s_m\, z|}{m^{1-1/n}} \int_m^{m+1} \frac{ds}{\left|s^{z+1/n}\right|}
\end{aligned}$$

Let $\sigma = \mathrm{Re}(z)$ and suppose $\sigma > 1 - 1/n$. $s_t = O(t^{1-1/n})$, so let C be an upper bound for $\left|s_t/t^{1-1/n}\right|$. We have

$$\begin{aligned}
\left|\sum_{m=1}^{t} \frac{a_m - \rho_{\mathfrak{m}}}{m^z}\right| &= C + C|z| \sum_{m=1}^{t-1} \int_m^{m+1} \frac{ds}{s^{\delta+1/n}} \\
&= C + C|z| \int_1^t \frac{ds}{s^{\delta+1/n}}
\end{aligned}$$

which is bounded. Thus,

$$\sum_{m=1}^{t} \frac{a_m - \rho_{\mathfrak{m}}}{m^z}$$

converges to an analytic function for $\mathrm{Re}(z) > 1 - 1/n$. Using the results of Section 5,

$$\zeta(z, \mathfrak{J}) = \left(\sum_{m=1}^{t} \frac{a_m - \rho_{\mathfrak{m}}}{m^z}\right) + \rho_{\mathfrak{m}} \zeta_{\mathbb{Q}}(z)$$

is analytic on $\{z : \mathrm{Re}(z) > 1 - 1/n\}$, except at $z = 1$ where it has a pole with residue $\rho_{\mathfrak{m}}$. The zeta function for the number field K satisfies

$$\zeta_K(z) = \sum_{\mathfrak{J} \in \mathrm{Cl}(K)} \zeta(z, \mathfrak{J}),$$

thus $\zeta_K(z)$ is analytic on $\{z : \mathrm{Re}(z) > 1 - 1/n\}$ except for a pole at 1 with residue $\rho_1 \left|\mathrm{Cl}(K)\right|$. Finally, if $\chi : \mathrm{Cl}_{\mathfrak{m}} \to \mathbb{C}^*$ is a nontrivial character, then

$$\begin{aligned} L_{\mathfrak{m}}(z,\chi) &= \sum_{\mathfrak{J} \in \mathrm{Cl}(K)} \chi(\mathfrak{J})\zeta(z,\mathfrak{J}) \\ &= \sum_{\mathfrak{J} \in \mathrm{Cl}(K)} \chi(\mathfrak{J}) \left(\sum_{m=1}^{t} \frac{a_m(\mathfrak{J}) - \rho_{\mathfrak{m}}}{m^z} \right) + \left(\sum_{\mathfrak{J} \in \mathrm{Cl}(K)} \chi(\mathfrak{J}) \right) \rho_{\mathfrak{m}} \zeta_{\mathbb{Q}}(z) \\ &= \sum_{\mathfrak{J} \in \mathrm{Cl}(K)} \chi(\mathfrak{J}) \left(\sum_{m=1}^{t} \frac{a_m(\mathfrak{J}) - \rho_{\mathfrak{m}}}{m^z} \right) \end{aligned}$$

using Proposition 3.2. Thus $L_{\mathfrak{m}}(z,\chi)$ is analytic on all of the region $\{z : \mathrm{Re}(z) > 1 - 1/n\}$.

8. The Universal Norm Index Inequality

An inequality that is useful in class field theory can be proven from the information about $\zeta_K(z)$ and $L_{\mathfrak{m}}(z,\chi)$ near $z = 1$ that we developed above. We will need some asymptotic notation: for analytic functions f and g, write $f \sim g \quad (z \to 1)$ to mean that f/g can be continued analytically to 1. Write $f \precsim g \quad (z \to 1)$ and $f \succsim g \quad (z \to 1)$ to mean that f/g is bounded above near 1, and bounded below near 1, respectively. Recall the series for the logarithm of $L_{\mathfrak{m}}$:

$$L_{\mathfrak{m}}(z,\chi) = \exp\left(\sum_{\mathfrak{p} \nmid \mathfrak{m}} \sum_{m} \frac{\chi(\mathfrak{p})^m}{m\mathbf{N}(\mathfrak{p})^{mz}} \right).$$

The sum of the second- and higher-order terms in the series converges for z near 1, thus

$$L_{\mathfrak{m}}(z,\chi) \sim \exp\left(\sum_{\mathfrak{p} \nmid \mathfrak{m}} \frac{\chi(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^z} \right) \quad (z \to 1).$$

Now, suppose L/K is a Galois extension, and $B \subseteq L$ the ring of integers. Let $\mathfrak{N}_L^{\mathfrak{m}} \subseteq K$ be the group of norms of fractional ideals in L that are coprime to $\mathfrak{m}$. We are interested in the subgroup of the ray class group,

$$\mathbf{P}^{\mathfrak{m}} \subseteq \mathfrak{N}_L^{\mathfrak{m}} \mathbf{P}^{\mathfrak{m}} \subseteq \mathbf{I}^{\mathfrak{m}}.$$

Theorem 8.1. *If $\mathfrak{m}$ is divisible by all the ramified primes of L/K, then*

$$(\boldsymbol{I}^{\mathfrak{m}} : \boldsymbol{P}^{\mathfrak{m}} \mathfrak{N}_L^{\mathfrak{m}}) \leq [L : K].$$

(The degree $(\boldsymbol{I}^{\mathfrak{m}} : \boldsymbol{P}^{\mathfrak{m}} \mathfrak{N}_L^{\mathfrak{m}})$ is called the norm index of K.)

Proof. Write $H = \mathbf{P}^{\mathfrak{m}} \mathfrak{N}_L^{\mathfrak{m}}$ for convenience. Consider the product

$$\prod_{\chi : \mathbf{I}^{\mathfrak{m}}/H \to \mathbb{C}^*} L_{\mathfrak{m}}(z,\chi).$$

Recall that $L_{\mathfrak{m}}(z,\chi)$ is analytic at 1 unless χ is trivial, in which case $L_{\mathfrak{m}}(z,\chi)$ has a simple pole at 1. Thus,

$$\frac{1}{z-1} \succsim \prod_{\chi} L_{\mathfrak{m}}(z,\chi) \quad (z \to 1).$$

Replacing $L_{\mathfrak{m}}(z,\chi)$ with the approximation given above,

$$\begin{aligned}
\frac{1}{z-1} &\succsim \exp\left(\sum_{\chi}\sum_{\mathfrak{p}\nmid\mathfrak{m}}\frac{\chi(\mathfrak{p})}{\mathbf{N}(\mathfrak{p})^z}\right)\\
&= \exp\left[\sum_{\mathfrak{J}\in\mathbf{I}^{\mathfrak{m}}/H}\left(\sum_{\chi}\chi(\mathfrak{J})\right)\sum_{\mathfrak{p}\in\mathfrak{J}}\frac{1}{\mathbf{N}(\mathfrak{p})^z}\right]\\
&= \exp\left[(\mathbf{I}^{\mathfrak{m}}:H)\sum_{\mathfrak{p}\in H}\frac{1}{\mathbf{N}(\mathfrak{p})^z}\right]
\end{aligned}$$

by Proposition 3.3.

Now, if $\mathfrak{p}$ is a prime in K, and $\{\mathfrak{q}_i\}_{i=1}^k$ are the primes lying above $\mathfrak{p}$ with residue field extension degrees $\deg\mathfrak{q}_i$, then $\mathbf{N}(\mathfrak{q}) = \mathbf{N}(\mathfrak{p})^{\deg\mathfrak{q}_i}$, so

$$\begin{aligned}
\sum_{j=1}^{k}\frac{1}{\mathbf{N}(\mathfrak{q}_j)^z} &= \sum_{\deg\mathfrak{q}_i=1}\frac{1}{\mathbf{N}(\mathfrak{p})^z} + \{\text{terms with exponent} \geq 2z\}\\
&\leq \frac{[L:K]}{\mathbf{N}(\mathfrak{p})^z} + \{\text{terms with exponent} \geq 2z\}
\end{aligned}$$

And every prime $\mathfrak{q}$ in B lies over a prime in H by definition. Thus, by ignoring terms with exponent $\geq 2z$ (which are asymptotically irrelevant near $z=1$) we find

$$\begin{aligned}
\frac{1}{z-1} &\succsim \exp\left((\mathbf{I}^{\mathfrak{m}}:H)\sum_{\mathfrak{p}\in H}\frac{1}{\mathbf{N}(\mathfrak{p})^z}\right)\\
&\succsim \exp\left(\frac{(\mathbf{I}^{\mathfrak{m}}:H)}{[L:K]}\sum_{\mathfrak{q}\subseteq B}\frac{1}{\mathbf{N}(\mathfrak{q})^z}\right)\\
&= \zeta_L(z)^{\frac{(\mathbf{I}^m:H)}{[L:K]}}\\
&\succsim \left(\frac{1}{z-1}\right)^{\frac{(\mathbf{I}^m:H)}{[L:K]}}
\end{aligned}$$

Thus $(\mathbf{I}^m : H) \leq [L:K]$, as desired. □

References

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