Math 254B, UC Berkeley, Spring 2002 (Kedlaya)
An "Abstract" Reciprocity Map

Reference. Milne VII.5; Neukirch VI.4, but only loosely.

In this lecture, we'll manufacture a canonical isomorphism $\mathrm{Gal}(L/K)^{\mathrm{ab}} \to C_K / \mathrm{Norm}_{L/K} C_L$ for any finite extension L/K of number fields, where C_K and C_L are the corresponding idele class groups. However, we won't yet know it agrees with our proposed reciprocity map, which is the product of the local reciprocity maps. We'll check that in the next lecture.

Cyclotomic extensions.

The cyclotomic extensions (extensions by roots of unity) of a number field play a role in class field theory analogous to the role played by the unramified extensions in local class field theory. This makes it essential to make an explicit study of them for use in proving the main results.

First of all, we should make a distinction that will come up repeatedly. The extension $\cup_n \mathbb{Q}(\zeta_n)$ of $\mathbb{Q}$ obtained by adjoining all roots of unity has Galois group $\widehat{\mathbb{Z}}^* = \prod_p \mathbb{Z}_p^*$. That group has a lot of torsion, since each $\mathbb{Z}_p^*$ contains a torsion subgroup of order $p-1$ (or 2, if $p=2$). If we take the fixed field for the torsion subgroup of $\mathbb{Z}^*$, we get a slightly smaller extension, which I'll call the *small cyclotomic* extension of $\mathbb{Q}$ and denote $\mathbb{Q}^{\mathrm{smcy}}$. Its Galois group is $\prod_p \mathbb{Z}_p = \widehat{\mathbb{Z}}$. For K a number field, define $K^{\mathrm{smcy}} = K\mathbb{Q}^{\mathrm{smcy}}$; then $\mathrm{Gal}(K^{\mathrm{smcy}}/K) \cong \widehat{\mathbb{Z}}$ as well, even if K contains some extra roots of unity.

The reciprocity map via abstract CFT.

First of all, we choose an isomorphism of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{smcy}}/\mathbb{Q})$ with $\widehat{\mathbb{Z}}$; our results are not going to depend on the choice. That gives a continuous surjection

$$d : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{smcy}}/\mathbb{Q}) \cong \widehat{\mathbb{Z}};$$

if we regard $\mathbb{Q}^{\mathrm{smcy}}/\mathbb{Q}$ as the "maximal unramified extension" of $\mathbb{Q}$, we can define the ramification index $e_{L/K}$ and inertia degree $f_{L/K}$ for any extension of number fields, by the rules

$$f_{L/K} = [L \cap \mathbb{Q}^{\mathrm{smcy}} : K \cap \mathbb{Q}^{\mathrm{smcy}}], \qquad e_{L/K} = \frac{[L:K]}{f_{L/K}}.$$

To use abstract class field theory to exhibit the reciprocity map, we need a "henselian valuation" $v : C_{\mathbb{Q}} \to \widehat{\mathbb{Z}}$, i.e., a homomorphism satisfying:

(i) $v(C_{\mathbb{Q}})$ is a subgroup Z of $\widehat{\mathbb{Z}}$ containing $\mathbb{Z}$ with $Z/nZ \cong \mathbb{Z}/n\mathbb{Z}$ for all $n \in \mathbb{N}$;

(ii) $v(\mathrm{Norm}_{K/\mathbb{Q}} C_K) = f_{K/\mathbb{Q}} Z$ for all finite extensions $K/\mathbb{Q}$.

Once we have that, our calculations from the last few lectures imply that the class field axiom is satisfied: for L/K cyclic,

$$\# H_T^0(\mathrm{Gal}(L/K), C_L) = [L:K], \qquad \# H_T^1(\mathrm{Gal}(L/K), C_L) = 1.$$

So then abstract class field theory will kick in.

We can make that valuation using Artin reciprocity for $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. Recall how that works: there is a canonical surjection

$$I_n \to (\mathbb{Z}/n\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) :$$

for p not dividing n, the ideal (p) goes to $p \in (\mathbb{Z}/n\mathbb{Z})^*$ and then to the automorphism $\zeta_n \mapsto \zeta_n^p$, which indeed does act as the p-th power map modulo any prime of $\mathbb{Q}(\zeta_n)$ above p.

That induces a homomorphism $I_{\mathbb{Q}} \to (\mathbb{Z}/n\mathbb{Z})^*$ as follows: given an idele α, pick $x \in \mathbb{Q}^*$ so that $\alpha_{\mathbb{R}}/x > 0$, and for each prime p with $p^e|n$, α/x has p-component congruent to 1 modulo p^e. Then map α/x to $(\mathbb{Z}/n\mathbb{Z})^*$ as follows:

$$\alpha/x \mapsto \prod_{\ell \not| n} \ell^{v_\ell(\alpha_\ell/x)}$$

This gives a well-defined map: if y is an alternate choice for x, then $x/y \equiv 1 \pmod{n}$ and $x/y > 0$, so the product on the right side is precisely x/y itself, and so is congruent to 1 in $(\mathbb{Z}/n\mathbb{Z})^*$.

We now have maps $I_{\mathbb{Q}} \to (\mathbb{Z}/n\mathbb{Z})^*$ which are easily seen to be compatible, so by taking inverse limits we get $I_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$. We define v by piping this map through the projection $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{smcy}}/\mathbb{Q})$ and then using our chosen isomorphism $\mathrm{Gal}(\mathbb{Q}^{\mathrm{smcy}}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}$.

Another way to say this: $I_{\mathbb{Q}}$ can be written as $\mathbb{Q}^* \times \mathbb{R}^* \times \widehat{\mathbb{Z}}^*$, and the map to $\widehat{\mathbb{Z}}^*$ is just projection onto the third factor! In particular, the map factors through $C_{\mathbb{Q}}$, and property (i) above is straightforward.

To check (ii), we need to do the same thing that we just did a bit more generally. For K now a number field, define the map

$$I_n \to (\mathbb{Z}/n\mathbb{Z})^* \supseteq \mathrm{Gal}(K(\zeta_n)/K),$$

where now I_n is the group of fractional ideals of K coprime to (n), by sending a prime $\mathfrak{p}$ first to its absolute norm. We then have to check that the result is always in the image of $\mathrm{Gal}(K(\zeta_n)/K)$, but in fact it must be: whatever the Frobenius of $\mathfrak{p}$ is, it sends ζ_n to a power of ζ_n congruent to $\zeta_n^{\mathrm{Norm}(\mathfrak{p})}$ modulo $\mathfrak{p}$. Since $\mathfrak{p}$ is prime to n, it's prime to the difference between any two powers of ζ_n, so the Frobenius of $\mathfrak{p}$ must in fact send ζ_n to $\zeta_n^{\mathrm{Norm}(\mathfrak{p})}$. This tells us first that the map above sends I_n to $\mathrm{Gal}(K(\zeta_n)/K)$ and second that it coincides with the Artin map.

From the First Inequality, we can deduce the following handy fact:

Proposition 1. *For L/K a finite abelian extension of number fields, the Artin map always surjects onto* $\mathrm{Gal}(L/K)$.

Proof. If the Artin map only hit the subgroup H of $\mathrm{Gal}(L/K)$, the fixed field M of H would have the property that all but finitely many primes of M split completely in L. We've already seen that this contradicts the First Inequality. $\square$

In particular, the Artin map $I_n \to \mathrm{Gal}(K(\zeta_n)/K)$ we wrote down above is surjective.

Using that, we can verify (ii): given a prime ideal $\mathfrak{p}$ of K, the Artin map of $K(\zeta_n)/K$ applied to it gives the same element of $(\mathbb{Z}/n\mathbb{Z})^*$ as the Artin map of $\mathbb{Q}$ applied to $\mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{p})$. And the Artin map of $K(\zeta_n)/K$ surjects onto $\mathrm{Gal}(K(\zeta_n)/K)$, which has index $[K \cap \mathbb{Q}(\zeta_n) : \mathbb{Q}]$ in $(\mathbb{Z}/n\mathbb{Z})^*$. This verifies (ii).

Thus lo and behold, we get from abstract class field theory a reciprocity isomorphism for any finite extension of number fields:

$$r'_{L/K} : C_K / \mathrm{Norm}_{L/K} C_L \xrightarrow{\sim} \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

These are compatible in the usual way, so we get a map $r'_K : C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Of course, we don't know what this map is, so we can't yet use it to recover Artin reciprocity. (That depended on the reciprocity map being the product of the local maps.) But at least we get the norm limitation theorem: if L/K is a finite extension and $M = L \cap K^{\mathrm{ab}}$, then $\mathrm{Norm}_{L/K} C_L = \mathrm{Norm}_{M/K} C_M$.

We do know one thing about this map: for "unramified" extensions L/K (i.e., $L \subseteq K^{\mathrm{smcy}}$), the "Frobenius" in $\mathrm{Gal}(L/K)$ maps to a "uniformizer" in C_K. That is, the element of $\mathrm{Gal}(L/K)$ coming from the element of $\mathrm{Gal}(K^{\mathrm{smcy}}/K)$ which maps to 1 under d_K is the element of C_K which maps to 1 under v_K. But we made v_K simply by mapping C_K to $\mathrm{Gal}(K^{\mathrm{smcy}}/K)$ via the Artin map and then identifying the latter with $\widehat{\mathbb{Z}}$ by the same identification we used to make d_K. Upshot: the choice of that identification drops out, and the reciprocity map coincides with the Artin map that we wrote down earlier.

To do next time: check that $r'_{L/K}$ agrees with the map that I called $r_{L/K}$, namely the product of the local reciprocity maps. (Remember, I need this in order to recover Artin reciprocity in general.)