Brauer Groups

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Abstract

I define the Brauer group of a field k as similarity classes of central simple algebras over k. Then I introduce non-abelean cohomology and use it to prove that the Brauer group is isomorphic to a certain cohomology group. Brauer groups show up in global class field theory.

1 Simple Algebras

A ring A is called a *k-algebra* if it contains a field k in its center and is a finite dimensional k-vectorspace. If A is a subalgebra of a k-algebra E, then the *centralizer* $C_E(A)$ of A in E is the set of elements of E which commute with all elements of A.

$Z(A) := C_A(A)$ is called the *center* of A. Its opposite A^{opp} is A with reversed multiplication.

Remark 1.1. *A k-algebra A is just a k-vectorspace V and an $x \in V \otimes_k V^* \otimes_k V^*$ which describes the multiplication in A: Given V and $x = \sum x_i \otimes \phi_i \otimes \psi_i$, A is V with the multiplication of two elements $a, b \in V$ defined as $a \cdot b = \sum \phi_i(a) \cdot \psi_i(b) \cdot x_i$.*

On the other hand, given A, one chooses V as the underlying vectorspace of A forgetting the multiplication, and after choosing a basis $e_1, \dots, e_n$ of V, choose ϕ_i as the projection to e_i and $x_{ij} = e_i \cdot e_j$. Then $x = \sum x_{ij} \otimes \phi_i \otimes \phi_j$ corresponds to the multiplication in V.

Whenever we talk about A-modules, we mean finitely generated left A-modules.

Definition 1.2. *An A-module V is* simple *if it is nonzero and it has no A-submodules besides 0 and V.*

Definition 1.3. *A k-algebra A is* simple *if its only two-sided ideals are 0 and A.*

Definition 1.4. *A k-algebra A is a* division algebra *if the units of A are $A \setminus \{0\}$.*

Example 1.5. *Let $M_n(A)$ denote the algebra of $n \times n$ matrices over an algebra A. For a division algebra D, $M_n(D)$ is simple.*

Theorem 1.6. *Let A be a simple k-algebra. Then A is isomorphic to $M_n(D)$ for some n and some division k-algebra D.*

Proof. Let S be a simple A-module (for example a minimal left ideal of A). By left-multiplication, we get a homomorphism $A \to E := \mathrm{End}_k(S)$ which is injective: since A is simple, its kernel, which is a two-sided ideal of A, must be 0 or A, but it is not A since $1 \mapsto 1$.

Now $C_E(A) = \mathrm{End}_A(S)$ which is a division algebra by Schur's Lemma, $A = C_E(C_E(A)) = \mathrm{End}_{C_E(A)}(S)$ by the Double Centralizer Theorem, and $\mathrm{End}_{C_E(A)}(S)$ is isomorphic to a matrix algebra over $C_E(A)^{\mathrm{opp}}$. □

Definition 1.7. *A k-algebra A is* central *if its center is k. It is* central simple *if it is central and simple.*

Lemma 1.8. *A k-algebra A is central simple if and only if it is isomorphic to $M_n(D)$ for some division algebra D with center k.*

Proof. Because A is simple, by theorem 1.6 it is isomorphic to some $M_n(D)$ with center k. On the other hand, the center of $M_n(D)$ is the set of all $d \cdot E_n$ where E_n is the unit matrix and d is in the center of D.

In the other direction, $M_n(D)$ is central simple because matrix algebras are simple. □

Remark 1.9. *In the lemma above, D is uniquely determined up to isomorphism.*

Theorem 1.10. *The tensor product of central simple k-algebras is central simple.*

Proof. Omitted. □

Let A and B be central simple k-algebras. They are called similar (write $A \sim B$) if $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$ for some m and n. Let $\mathrm{Br}(k)$ be the set of central simple k-algebras modulo the equivalence relation $\sim$. This is an equivalence relation. $A \otimes_k B$ is again central simple by theorem 1.10, and if $A \sim A'$ and $B \sim B'$, then $A \otimes_k B \sim A' \otimes_k B'$, so one gets a well-defined multiplication on $\mathrm{Br}(k)$. Since $A \otimes_k M_n(k) \sim A$, $M_n(k)$ is a neutral element for any n, and since $A \otimes_k A^{\mathrm{opp}} \sim M_n(k)$, $\mathrm{Br}(k)$ is a group.

Definition 1.11. *The group of similarity classes of central simple k-algebras with the above defined multiplications is called the* Brauer group *of k, denoted by* $\mathrm{Br}(k)$.

Remark 1.12. *There is a bijection between the elements of* $\mathrm{Br}(k)$ *and the division algebras D with center k, mapping D to the element represented by some $M_n(D)$.*

Proof. Since $M_n(D) \otimes_k M_m(k) \cong D \otimes_k M_n(k) \otimes_k M_m(k) \cong D \otimes M_{nm}(k) \cong M_{nm}(D)$, all the representatives of an element of $\mathrm{Br}(k)$ have the same underlying division algebra D. And $M_n(D)$ and $M_m(D)$ represent the same element because $M_n(D) \otimes_k M_m(k) \cong M_{nm}(D) \cong M_m(D) \cong M_n(k)$. □

Proposition 1.13. *Let A be a central simple k-algebra, and let L/k be a field extension. Then $A \otimes_k L$ is a central simple L-algebra.*

Proof. This is true because the tensor product of a simple and a central simple algebra is simple, and because the center of the tensor product is the tensor product of the centers. □

Let L/k be a field extension. Define a map $\mathrm{Br}(k) \to \mathrm{Br}(L)$ by $A \mapsto A \otimes_k L$. It is well-defined because of $(A \otimes_k M_n(k)) \otimes_k L \cong A \otimes_k M_n(L)$, and it is a homomorphism because of $(A \otimes_k L) \otimes_L (A' \otimes_k L) \cong (A \otimes_k A') \otimes_k L$, using the associativity of the tensor product.

Definition 1.14. *Let* $\mathrm{Br}(L/k)$ *be the kernel of the above defined map* $\mathrm{Br}(k) \to \mathrm{Br}(L)$. *An element of* $\mathrm{Br}(k)$ *(and any central simple k-algebra A that represents it) is* split *by L if it is in* $\mathrm{Br}(L/k)$, *i.e. if $A \otimes_k L$ is a matrix algebra over L.*

2 Non-abelian cohomology

In this chapter we define the cohomology of an arbitrary (not necessarily abelian) group A on which a group G acts on the left, i.e. a G-module. We define $H^i(G, A)$ only for $i = 0, 1$ directly without using resolutions, similar to the direct interpretation for these groups in the case that A is abelian.

Let $H^0(G, A) = A^G$, the elements of A invariant under the action of G. Further let $H^1(G, A) = C^1/\sim$ where C^1 is the set of maps $\phi : G \to A$ satisfying $\phi(gh) = \phi(g) \cdot g(\phi(h))$ and $\phi \sim \psi :\Leftrightarrow \phi(g) = a^{-1} \cdot \psi(g) \cdot g(a)$ for some $a \in A$. This is an equivalence relation.

Now $H^1(G, A)$ is not a group as in the abelian case, but we have a distinguished element, the identity map, which makes it a *pointed set*, and a map of pointed sets X and Y must send the distinguished element of X to the distinguished element of Y. We define the *kernel* of this map as the preimage of the the distinguished element of Y. Now we can talk about exact sequences of pointed sets.

As in the abelian case, given a group homomorphism $f : A \to B$ which commutes with the operation of G, we get maps $f_i : H^i(G, A) \to H^i(G, B)$ for $i = 1, 2$.

Given an exact sequence $1 \to A \to B \to C \to 1$ of G-modules, we get a *coboundary* map $\delta : H^0(G, C) \to H^1(G, A)$ as follows: Given $c \in C^G$, choose $b \in B$ such that $b \mapsto c$. Then for each $g \in G$, $b^{-1}g(b)$ is the image of an element of A and call it $\phi(g)$. Now let $\delta(c) = \phi$. This defines an element of $H^1(G, A)$ which is independent of all the choices made. As all the other definition, this definition coincides with the usual definition if the groups are abelian, and the proofs work exactly as in that case; we just have to take care of not switching things unnecessarily around.

Proposition 2.1. *If $1 \to A \to B \to C \to 1$ is an exact sequence of G-modules, then*

$$1 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \\ \to H^1(G, A) \to H^1(G, B) \to H^1(G, C)$$

is an exact sequence of pointed sets.

If A is contained in the center of B, then A is abelian, and we have $H^2(G, A)$, and we can define $\Delta : H^1(G, C) \to H^2(G, A)$ as follows.

Using the explicit description of $H^i(G, A)$ given in [2], a cocycle for $H^2(G, A)$ is given by a function $G^3 \to A$ invariant under an action of G satisfying certain relations. Similar to the case $H^1(G, A)$, it is determined by its values with the identity of G as the first argument. Therefore it can be described as a function $\phi : G^2 \to A$ satisfying $g(\phi(h, k)) \cdot \phi(g, hk) = \phi(gh, k) \cdot \phi(g, h)$.

Let c be a cocycle of $H^1(G, C)$. Then for each $g \in G$, choose a $b(g) \in B$ such that $b(g) \mapsto c(g)$. Now for each $h \in G$, $b(g) \cdot g(b(h)) \cdot b(gh)^{-1} \in A$, and we can define this as $\phi(g, h)$ to get a cocycle ϕ of $H^2(G, A)$. Define Δ by mapping c to ϕ. This definition is independent of all the choice made.

Proposition 2.2. *Under the conditions of prop. 2.1, with A contained in the center of B, the sequence stays exact if we add $H^2(G, A)$ at the end using Δ.*

3 The Brauer Group and Cohomology

Let L/k be a finite Galois extension of fields, and let $G = \mathrm{Gal}(L/k)$ and $H^2(L/k) = H^2(G, L^*)$. We will prove that $H^2(L/k) \cong \mathrm{Br}(L/k)$.

Let $\mathrm{Br}_n(L/k)$ be the set of elements of $\mathrm{Br}(L/k)$ (see def. 1.14) which are represented by algebras A such that $A \otimes_k L \cong M_n(L)$. Then $\mathrm{Br}(L/k) = \bigcup \mathrm{Br}_n(L/k)$.

Consider the algebras representing elements of $\mathrm{Br}_n(L/k)$ as pairs (V, x) with V a n^2-dimensional k-vectorspace and $x \in V \otimes_k V^* \otimes_k V^*$ as in remark 1.1.

(V, x) and (V', x') are called k-isomorphic if there is a vectorspace-isomorphism $f : V \to V'$ such that $f(x) = x'$ where $f(x) = \sum x_i \otimes \phi_i \circ f^{-1} \otimes \psi_i \circ f^{-1}$ for $x = \sum x_i \otimes \phi_i \otimes \psi_i$. Such an f exists if and only if the corresponding k-algebras are isomorphic.

The fact that $A \otimes_k L \cong M_n(L)$ corresponds to $(V \otimes_k L, x \otimes 1)$ being L-isomorphic to $(M_n(L), x_0)$ where x_0 describes the standard multiplication of matrices. And $(V, x) \sim (V', x')$ if and only if they are k-isomorphic.

Let $C_n(L) = \mathrm{Aut}_L(M_n(L))$, the group of L-algebra automorphisms of $M_n(L)$. Then G acts on $M_n(L)$ component-wise, and for $g \in G$ and $\phi : M_n(L) \to M_n(L)$, we define $g(\phi) = g \circ \phi \circ g^{-1}$, which is an L-linear map, so we get an action of G on $C_n(L)$.

For $g \in G$, $[(V, x)] \in \mathrm{Br}_n(L/k)$ and $f : M_n(L) \to V \otimes_k L$ an L-isomorphism mapping $(V \otimes_k L, x \otimes 1)$ to $(M_n(L), x_0)$, let $\theta_f(g) = f^{-1} \circ$

$g \circ f \circ g^{-1}$. Then $\theta_f(g) : M_n(L) \to M_n(L)$ is an isomorphism because f is, and it is L-linear, so $\theta_f(g) \in C_n(L)$.

Now $\theta_f : G \to C_n(L)$ is a 1-cocycle: $\theta_f(gg') = f^{-1} \circ gg' \circ f \circ (gg')^{-1} = (f^{-1} \circ g \circ f \circ g^{-1}) \circ (g \circ (f^{-1} \circ g' \circ f \circ g'^{-1}) \circ g^{-1}) = \theta_f(g) \circ g(\theta_f(g'))$.

If one chooses a different f' instead of f, then $f' = \phi \circ f$ for some $\phi \in C_n(L)$, so $\theta_{f'}(g) = (\phi f)^{-1} \circ g \circ (\phi f) \circ g^{-1} = (f^{-1}\phi^{-1}f) \circ (f^{-1} \circ g \circ f \circ g^{-1}) \circ (g \circ (f^{-1}\phi f) \circ g^{-1}) = (f^{-1}\phi f)^{-1} \circ \theta_f(g) \circ g(f^{-1}\phi f)$, so $\theta_{f'}$ and θ_f are 1-cohomologous.

If one chooses (V', x') representing the same element of $\mathrm{Br}_n(L/k)$ as (V, x), then $f' = \phi \circ f$ for some $\phi : V \to V'$, and the same computation as above shows that θ_f and $\theta_{f'}$ are 1-cohomologous.

Therefore one gets a well-defined map $\theta : \mathrm{Br}_n(L/k) \to H^1(G, C_n(L))$ defined by $(V, x) \mapsto \theta_f$ as above.

Proposition 3.1. *The map $\theta : \mathrm{Br}_n(L/k) \to H^1(G, C_n(L))$ as above is bijective.*

Proof. If (V, x) with f and (V', x') with f' give $\theta_f = \theta_{f'}$, then $f^{-1} \circ g \circ f \circ g^{-1} = f'^{-1} \circ g \circ f' \circ g^{-1}$, so $g^{-1} \circ f'f^{-1} \circ g = f'f^{-1}$, i.e. $g(f'f^{-1}) = f'f^{-1}$, and therefore the L-isomorphism $f'f^{-1}$ is a k-isomorphism, so $[(V, x)] = [(V', x')] \in \mathrm{Br}_n(L/k)$, and θ is injective. The element represented by $M_n(L)$ is mapped to 0 because in this case, f is the identity, so θ_f is zero.

Now let $\phi : G \to C_n(L)$ be an arbitrary 1-cocycle. Since $H^1(G, \mathrm{GL}_{n^2}(L))$ is trivial (by exercise 3 of [1]) and $C_n(L) \subset \mathrm{GL}_{n^2}(L)$, there is an L-automorphism f of $M_n(L)$ such that $\phi(g) = f^{-1} \circ g(f)$ for all $g \in G$. Let $x' = f(x_0)$. Then $g(x') = g(f(x_0)) = g(f)(g(x_0)) = g(f)(x_0) = (f \circ \phi(g))(x_0) = f(\phi(g)(x_0)) = f(x_0) = x'$ where x_0 corresponding to the standard matrix multiplications is independent of base changes and the operation of G. Therefore x' is defined over k, and $[(M_n(k), x')] \in \mathrm{Br}_n(L/k)$ maps to $[\phi] \in H^1(G, C_n(L)$, so θ is also surjective. □

Because every automorphism of $M_n(L)$ is inner, the map $\mathrm{GL}_n(L) \to C_n(L)$ mapping $\phi \in \mathrm{GL}_n(L)$ to the conjugation by ϕ is surjective, and since the center of $GL_n(L)$ is just $\{L^* \cdot E_n\}$ where E_n is the unit matrix of $M_n(L)$, the sequence $1 \to L^* \to \mathrm{GL}_n(L) \to C_n(L) \to 1$ is exact.

By the long exact sequence in cohomology, this gives a map $\Delta_n : H^1(G, C_n(L)) \to H^2(G, L^*)$, and $\delta_n = \Delta_n \circ \theta : Br_n(L/k) \to H^2(G, L^*)$.

The different δ_n are compatible: For $C \in Br_n(L/k)$, $\delta_n(C) = 0$ if and only if $\Delta_n(C) = 0$ since θ is bijective, and this is true if and only if C

is represented by a matrix algebra because θ is a bijection mapping this element to zero by 3.1, and Δ_n is injective because the preceding term in the long exact sequence is $H^1(G, \mathrm{GL}_n(L)) = 0$ (again by exercise 3 of [1]). And $\delta_n(C) + \delta_{n'}(C') = \delta_{nn'}(C \otimes_k C')$ by an easy computation. Therefore the $delta_n$ give an injective homomorphism $\delta : \mathrm{Br}(L/k) \to H^2(L/k)$.

Theorem 3.2. *The map $\delta : \mathrm{Br}(L/k) \to H^2(L/k)$ is an isomorphism.*

Proof. Because δ is injective and because of prop. 3.1, it is enough to show that Δ_n is surjective for $n = [L : k]$.

Let $a : G \times G \to L^* \subset \mathrm{GL}_n(L)$ be an arbitrary cocycle. Let V be the L-vectorspace with basis $\{e_h, h \in G\}$, and let p_g be the automorphism of V defined by $p_g(e_h) = a(g, h) \cdot e_{gh}$. Then $p_s(s(p_t)(e_u)) = a(s, tu) \cdot s(a(t, u)) \cdot (e_{stu})$ and $a(s, t) \cdot p_{st}(e_u) = a(s, t) \cdot a(st, u) \cdot e_{stu}$, and since $a(s, t) \cdot a(st, u) = a(s, tu) \cdot s(a(t, u))$, we get $a(s, t) = p_s(s(p_t)(p_{st}^{-1}))$, so a is in the image of Δ_n. □

Proposition 3.3. *A k-algebra A is central simple if and only if $A \otimes_k \bar{k} = M_n(\bar{k})$ for some n, where $\bar{k}$ is the algebraic closure of k.*

Proof. See Bourbaki, Algebra, Chapter VIII. □

Proposition 3.4. *A k-algebra A is central simple if and only if $A \otimes_k L = M_n(L)$ for some n and some finite Galois extension L/k.*

Proof. Choose a basis $e_1, \ldots, e_{n^2}$ of A, and let $A_1, \ldots, A_{n^2}$ be the images of $e_i \otimes \bar{k}$ in $M_n(\bar{k})$, using the isomorphism from prop. 3.3. Let L be a finite Galois extension containing the entries of the A_i. Then the isomorphism above induces an isomorphism $A \otimes_k L \to M_n(L)$. □

Theorem 3.5. $\mathrm{Br}(k) \cong H^2(\bar{k}/k)$.

Proof. The isomorphism $\mathrm{Br}(L/k) \to H^2(L/k)$ are compatible with the maps $\mathrm{Br}(L/k) \to \mathrm{Br}(L'/k)$ and $H^2(L/k) \to H^2(L'/k)$ if $L'/L/k$ are finite Galois extensions. Because of prop. 3.3, $Br(\bar{k})$ is trivial, so $0 \to \mathrm{Br}(\bar{k}/k) \to \mathrm{Br}(k) \to \mathrm{Br}(\bar{k})$ gives $\mathrm{Br}(k) \cong \mathrm{Br}(\bar{k}/k)$, which is the limit of the $\mathrm{Br}(L/k)$ for L a finite Galois extension of k because of prop. 3.4, and $H^2(\bar{k}/k)$ is the limit of the $H^2(L/k)$. □

Example 3.6. *Let $k = \mathbb{R}$. Then $\bar{\mathbb{R}} = \mathbb{C}$, and $\mathrm{Br}(\mathbb{R}) = H^2(G, \mathbb{C}^*)$, where $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order 2. Hence $H^2(G, \mathbb{C}^*) = H^0_T(G, \mathbb{C}^*) =$*

$(\mathbb{C}^*)^G/\operatorname{Norm}_G(\mathbb{C}^*) = \mathbb{R}^*/\mathbb{R}^+ = \{\pm 1\}$. *The identity element of* $\operatorname{Br}(\mathbb{R})$ *is represented by the trivial central simple* $\mathbb{R}$*-algebra* $\mathbb{R}$ *itself.*

The nontrivial element is represented by the quaternions H*:* $\{1, i, j, k\}$ *is a basis of* H *as an* $\mathbb{R}$*-vectorspace, and the multiplication is defined by* $i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. H *is simple because it is a division algebra, and its center is* $\mathbb{R}$ *because every other element does not commute with at least one of* i, j *or* k. H *represents the nontrivial element in* $\operatorname{Br}(\mathbb{R})$ *because of the remark 1.12 about the bijection between the divison algebras with center* k *and the elements of the Brauer group (*H *is non-commutative, so not isomorphic to* $\mathbb{R}$*).*

Example 3.7. *For a local field* k, $\operatorname{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$ *by theorem 5 of [3], where the isomorphism is given by the local invariant map* inv_k.

4 Brauer Groups in Class Field Theory

In this section, k always denotes a number field. First, we want to prove that every element of the Brauer group k splits in some cyclic cyclotomic extension of k. We need two lemmas.

Lemma 4.1. *The map* $\operatorname{Br}(k) \to \bigoplus_v \operatorname{Br}(k_v)$ *is injective, where the direct sum goes over all the places* v *of* k.

Proof. This is proved at the beginning of the proof of theorem 4.4. □

Lemma 4.2. *Let* S *be a finite set of primes of* k*. Then for any* $m \in \mathbb{N}$*, there exists a cyclic cyclotomic extension* L/k *such that* m *divides* $[L_v : K_v]$ *for all* $v \in S$.

Proof. For an arbitrary number field, it is enough to find such an extension of $\mathbb{Q}$ with $m \cdot [k : \mathbb{Q}]$ instead of m.

So assume $k = \mathbb{Q}$. For q a prime, let ζ be a primitive q^rth root of unity. Then $\operatorname{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q}) \cong (\mathbb{Z}/q^r\mathbb{Z})^*$, which contains a quotient group of order q^s with $s \geq r - 3$, so s becomes arbitrarily large for large r. Let $L(q^r)$ be the subextension of $\mathbb{Q}[\zeta]/\mathbb{Q}$ with this Galois group. It is a cyclic cyclotomic extension of $\mathbb{Q}$.

Using theorems II.7.12 and II.7.13 of [8], we see that $[\mathbb{Q}_p[\zeta] : \mathbb{Q}] \to \infty$ as $r \to \infty$. Therefore $[L(q_r)_p : \mathbb{Q}_p]$ is an arbitrarily large power of q for large r.

Let $q_1, \ldots, q_s$ be the distinct primes dividing m, and let $L = L(q_1^{r_1}) \cdots L(q_s^{r_s})$. Then L is cyclic because it is the product of cyclic groups of pairwise coprime order, and after choosing the r_i large enough, m divides $[L_p : \mathbb{Q}_p]$ for all the finitely many $p \in S$. □

Theorem 4.3. *Let k be a number field. Then any $\alpha \in \mathrm{Br}(k)$ maps to $0 \in \mathrm{Br}(L)$ for some cyclic cyclotomic extension L/k (depending on α).*

Proof. Let (α_v) be the image of α in $\bigoplus \mathrm{Br}(k_v)$. Then almost all α_v are zero, and let m be the common denumerator of the nonzero $\mathrm{inv}_v(\alpha_v) \in \mathbb{Q}/\mathbb{Z}$. Let L be as in the lemma above.

Then $\mathrm{inv}_w(\alpha_w) = [L_w : k_v] \cdot \mathrm{inv}_v(\alpha_v) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all valuations w of L, i.e. the image of α in $Br(L_w)$ is zero for all w. Because $\mathrm{Br}(L)$ injects into $\bigoplus \mathrm{Br}(L_w)$, the image of α in $\mathrm{Br}(L)$ is zero. □

Theorem 4.4. (Fundamental exact sequence) $0 \to B(k) \to \bigoplus_v B(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$ *is exact, where the direct sum goes over all places v of k.*

Proof. Let L/k be a finite Galois extension with Galois group G. In [4] we defined $C_L = I_L/L^*$ where I_L are the ideles and C_L is the ideles class group. This gives us a long exact sequence containing $H^1(G, C_L) \to H^2(G, L^*) \to H^2(G, I_L) \to H^2(G, C_L)$ where $H^1(G, C_L) = 0$ by [5], $H^2(G, L^*) = \mathrm{Br}(L/k)$ and $H^2(G, I_L) = \bigoplus H^2(G_v, L_v^*) = \bigoplus \mathrm{Br}(L_v/k_v)$ by Prop. 1 of [5]. Therefore

$$0 \to \mathrm{Br}(L/k) \to \bigoplus \mathrm{Br}(L_v/k_v) \to H^2(G, C_L)$$

is exact. Let $H^2(G, C_L)'$ be the image of the last map.

By passing to the limit for all L, the first terms of this sequence give lemma 4.1.

The local invariant map gives the isomorphism $\mathrm{inv}_v : \mathrm{Br}(L_v/k_v) \to \frac{1}{n_v}\mathbb{Z}/\mathbb{Z}$, and summing these, we get a surjective map $\bigoplus \mathrm{Br}(L_v/k_v) \to \frac{1}{n_0}\mathbb{Z}/\mathbb{Z} \subset \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, where $n_0 = \mathrm{lcm}(n_v)$.

By Theorem VII.8.1 in [6], we get a (not necessarily exact) complex $0 \to \mathrm{Br}(L/K) \to \bigoplus(L_v/K_v) \to \frac{1}{n_0}\mathbb{Z}/\mathbb{Z}$, so together with the exact sequence above, this gives a map $\phi : H^2(G, C_L)' \to \frac{1}{n_0}\mathbb{Z}/\mathbb{Z}$.

Suppose $n_0 = n$. Then ϕ is an isomorphism since it is a surjective map with $\#H^2(G, C_L)' \leq \#H^2(G, C_L) \leq n$. So $H^2(G, C_L)' = H^2(G, C_L)$, both with order n, and $0 \to \mathrm{Br}(L/k) \to \bigoplus \mathrm{Br}(L_v/k_v) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to 0$ is exact.

If L/k is cyclic, then $n_0 = n$ can be proved using the Artin map: Let $\mathfrak{m}$ be a formal product of places containing the infinite and ramified ones.

Then the Artin map $I_k^{\mathfrak{m}} \to G$ maps $\mathfrak{p} = \mathfrak{p}_v \mapsto \mathrm{Frob}_{\mathfrak{p}}$, which has order $f_{\mathfrak{p}}$, and $f_{\mathfrak{p}} = n_v$ since $\mathfrak{p}$ is unramified. So the image of the Artin map has order $n_0 = \mathrm{lcm}(n_v) \leq n$. But G has order n since L/k is cyclic, and the Artin map is surjective, so $n = n_0$.

By passing to the direct limit, we get $0 \to \mathrm{Br}(\mathbb{Q}^{\mathrm{cyc}}k/k) \to \bigoplus \mathrm{Br}((\mathbb{Q}^{\mathrm{cyc}}k)_v/k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$, and since $\mathrm{Br}(\mathbb{Q}^{\mathrm{cyc}}k/k) = \mathrm{Br}(k)$ by theorem 4.3 and $\mathrm{Br}((\mathbb{Q}^{\mathrm{cyc}}k)_v/k_v) = \mathrm{Br}(k_v)$, the theorem is true. $\square$

References

[1] K. Kedlaya, *Kummer Theory*, Handout for Math254B, Spring 2002.

[2] K. Kedlaya, *The Cohomology of Finite Groups II: Concrete Nonsense*, Handout for Math254B, Spring 2002.

[3] K. Kedlaya, *Overview of Local Class Field Theory*, Handout for Math254B, Spring 2002.

[4] K. Kedlaya, *Adeles and Ideles*, Handout for Math254B, Spring 2002.

[5] K. Kedlaya, *Cohomology of I: The First Inequality*, Handout for Math254B, Spring 2002.

[6] J. Milne, *Class Field Theory*, www.jmilne.org, v3.1, 1997.

[7] J.-P. Serre, *Local Fields*, Springer Verlag, 1979.

[8] J. Neukirch, *Algebraische Zahlentheorie*, Springer Verlag, 1992